1.4.3 Menger universal curve

The *Menger universal curve* M_1^3 is the subset of the unit cube I^3 whose projections onto faces of the cube are the Sierpiński carpets, i.e.,

$$M_1^3 = \{ (x, y, z) \in I^3 : (x, y) \in M_1^2, \quad (y, z) \in M_1^2, \quad (x, z) \in M_1^2 \}$$

[Menger 1926]. Other descriptions can be found in 1.4.1 (see also [Mayer et al. 1986, pp. 5–6, 8–9]). See Figure A.

1. The following statements are equivalent:
   (a) X is homeomorphic to M_1^3;
   (b) X is a locally connected curve with no local cut points and no planar open nonempty subsets [Anderson 1958, Theorem XII, p. 13];
   (c) X is a homogeneous locally connected curve different from a simple close curve [Anderson 1958, Theorem XIII, p. 14];
   (d) X is a locally connected curve with the disjoint arcs property (see Property 3 in 1.4.1);
   (e) X is a locally connected curve and each arc in X is approximately non-locally-separating arc and has empty interior in X [Krupski et al. XXXXa, Theorem 3, p. 86].

2. M_1^3 is universal in the class of all metric separable spaces of dimension ≤ 1 [Menger 1926].

3. Z-setsZ-set in M_1^3 coincide with non-locally-separating closed subsets of M_1^3.

   If Z is a separable metric space of dimension ≤ 1, then Z can be embedded as a non-locally-separating subset of M_1^3 [Mayer et al. 1986, Theorem 6.1, p. 42].

4. If Z is a closed subset of a metric space X, $\dim(X \setminus Z) \leq 1$ and $f : Z \to M_1^3$ is a continuous mapping with non-locally separating image $f(Z)$, then f can be extended to a map $g : X \to M_1^3$ such that $g|X \setminus Z$ is an embedding into $M_1^3 \setminus f(Z)$ [Mayer et al. 1986, Theorem 6.4, p. 44].

5. Every continuous surjection h between non-locally separating closed subsets Z, Z' of M_1^3 extends to a mapping h^* of M_1^3 such that $h^*|M_1^3 \setminus Z$ is a homeomorphism onto $M_1^3 \setminus Z'$ [Mayer et al. 1986, Corollary 6.5, p. 44]; moreover, for every $\epsilon > 0$ there exists $\delta > 0$ such that if h is a δ-homeomorphism, then h^* can be taken as an ϵ-homeomorphism (see Property 4b in 1.4.1).

6. For each locally connected continuum X, there exist open surjections $f : M_1^3 \to X$ and $g : M_1^3 \to X$ such that $f^{-1}(x)$ is homeomorphic to M_1^3 and $g^{-1}(x)$ is a Cantor set, for any $x \in X$ [Wilson 1972].