Examples in Continuum Theory

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November 29, 2001

Contents

Chapter 0

PRELIMINARIES

All spaces considered here are assumed to be metric (if otherwise is not explicitly said) and all mappings are assumed to be continuous. Concepts and notions undefined in the text are used according to their meaning known from the literature. To aviod any doubt or misunderstanding, the reader can find the needed definition or explanation in Appendix A, where the items are alphabetically ordered.

The following notation will be used. The symbol $\mathbb{R}$ stands for the space of real numbers, equipped with the natural topology. Thus $\mathbb{R}^2$ denotes the plane (usually supplied with a Cartesian coordinate system); equivalently, the plane can be denoted by $\mathbb{C}$ when considered as the set of all complex numbers. The closed unit interval $[0,1]$ of reals is denoted by $\mathbb{I}$, and $\mathbb{N}$ means the set of all positive integers.

Chapter 1

LOCALLY CONNECTED CONTINUA

1.1 Basic examples: an arc, a circle, n-cube $\mathbb{I}^n$, Hilbert cube

1.2 Graphs

(a) Elementary examples: an arc, a circle, a noose, a figure eight, a θ-curve

(b) Acyclic graphs = trees.

(c) planability of graphs; the Kuratowski graphs $K_{3,3}$, K_5; Whyburn's example of an open map from a planar onto a nonplanar graph

(d) Problem of Königsberg bridges; Euler's solution

1.3 Dendrites

A dendrite is a locally connected continuum containing no simple closed curve. By the order of a point p in a dendrite X, writing $\operatorname{ord}(p, X)\operatorname{ord}(p, X)$, we mean the Menger-Urysohn order, see [Kuratowski 1968, §51, I, p. 274], or equivalently, the order in the classical sense, i.e., the number of arcs emanating from p and disjoint out of p (see [Charatonik 1962, p. 229] and [Lelek 1961, p. 301]).

In this chapter the term of an end point of a continuum is always used in the sense of a point of (Menger-Urysohn) order 1, i.e., p is an end point of X provided that $\operatorname{ord}(p, X) = 1$. For dendrites this concept coincides with one of an end point in the classical sense, but not with the notion of an end point of an arc-like continuum as defined e.g. in [Bing 1951, p. 660].

Given a dendrite X, we denote by $E(X)$ the set of all end points of X and by $R(X)$ the set of all its ramification points (i.e., points of order at least 3). Various structural as well as mapping characterizations of dendrites are collected in [Charatonik et al. 1998, Theorems 1.1 and 1.2, p. 228 and 230, respectively]. See also [Charatonik et al. 2000].

1.3.1 Preliminary properties

The following properties of dendrites are known (see e.g. [Nadler 1992, Chapter 10, p. 165], [Kuratowski 1968, §51, VI, p. 300] or [Whyburn 1942, p. 88]; compare also [Charatonik et al. 1998, Theorems 1.3 and 1.5, p. 233]).

1. Every subcontinuum of a dendrite is a dendrite (thus each dendrite is hereditarily locally connected).

2. Every connected subset of a dendrite is arcwise connected.

3. In any dendrite X the order of a point p is equal to the number of components of $X \setminus \{p\}$.

4. The set of all end points of a dendrite is 0-dimensional.

5. The set of all end points of a dendrite is a G_δ-set.

6. The set of all ordinary points (i.e., points of order 2) of a dendrite X is dense in X.

7. Every dendrite has at most countably many ramification points.

8. If m is the order of a ramification point in a dendrite, then $m \in \{3, 4, \ldots, \omega\}$.

9. No dendrite contains points of order $\aleph_0$ or $\mathfrak{c}$ (i.e., each dendrite is a regular continuum).

10. In any dendrite the set of all its end points is dense if and only if the set of all its ramification points is dense.

11. Dendrites are absolute retracts. In fact, the class of all 1-dimensional absolute retracts coincide with the class of dendrites, [Borsuk 1967, Corollary 13.5, p. 138].

12. A continuum X is a dendrite if and only if for every compact space (continuum) Y and for every light confluent mapping $f : Y \to f(Y)$ such that $X \subset f(Y)$ there is a copy X' of X in Y for which the restriction $f|X' : X' \to X$ is a homeomorphism, [Charatonik et al. XXXXb].

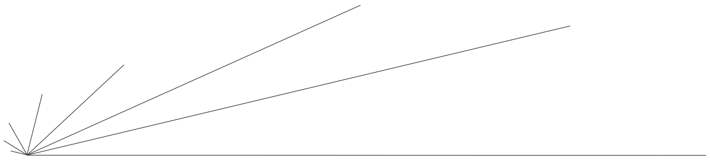


Figure 1.3.3: (A) locally connected fan F_ω

1.3.2 Simple examples

The simplest examples of dendrites are the arc and n-ods for $n \in \{3, 4, \dots\}$. All they are trees. By a tree we mean a dendrite with finitely many end points, or equivalently, a dendrite containing no points of order ω and having the set of all ramification points finite.

1.3.3 The locally connected fan

Another simple example is the *locally connected fan* F_ω. It is defined as the union of countably many straight line segments of length tending to 0, emanating from a point p and disjoint out of p. Thus F_ω is the dendrite with only one ramification point whose order is ω.

The following mapping properties of F_ω are known.

1. F_ω is universal in the class of all n-ods for $n \in \mathbb{N}$.

2. Each open image of F_ω is homeomorphic to F_ω, [Charatonik et al. 1990, Proposition 9.4, p. 42].

3. A confluent mapping defined on F_ω is open if and only if it is light, [Charatonik et al. 1990, Theorem 9.6, p. 42].

1.3.4 The locally connected combs

The next two examples are defined in [Arévalo et al. 2001, p. 3]. Let $\overline{pq}$ stand for the straight line segment from p to q in the plane. In the plane, let $a = (0, 0), a_n = (1/n, 1/n), b_n = (1/n, 0)$ for each $n \in \mathbb{N}$, and $c = (-1, 0)$. Define

$$W_R = \overline{ab_1} \cup \bigcup \{\overline{a_n b_n} : n \in \mathbb{N}\} \quad \text{and} \quad W = \overline{ca} \cup W_R.$$

The dendrites W_R and W are called the *comb* W_R and the *comb* W , or (generally) *locally connected combs* .

See Figures A–B.

The dendrites F_ω, W_R and W are exploited in the following characterizations.

1. A dendrite is a tree if and only if it contains neither a copy of F_ω nor of W_R, [Arévalo et al. 2001, Theorem 3.1, p. 3].

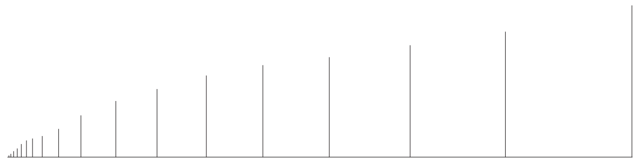


Figure 1.3.4: (A) comb W_R

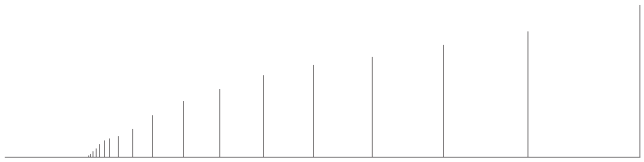


Figure 1.3.4: (B) comb W

2. A dendrite has the set of all its end points closed if and only if it contains neither a copy of F_ω nor of W, [Arévalo et al. 2001, Corollary 5.4, p. 11].

1.3.5 Universal dendrites

As it was mentioned in Property 1 in 1.3.3 the locally connected fan is universal in the class of all n-ods, where $n \in \mathbb{N}$. Now we recall the construction of the Ważewski universal dendrite D_ω and some other dendrites that are universal in various classes of these curves.

1.3.6 Ważewski universal dendrite

By the *Ważewski universal dendrite* D_ω we mean a dendrite D_ω such that each ramification point of D_ω is of order ω and for each arc $A \subset D_\omega$ the set of ramification points of D_ω which belong to A is dense in A. Its construction, known from [Whyburn 1942, Chapter K, p. 137] (compare also [Menger 1932, Chapter X, Section 6, p. 318]), is the following.

Let $X_1^{(\omega)} = F_\omega$. At the midpoint c of each maximal free arc contained in $X_1^{(\omega)}$ (obviously the arc is a straight line segment) attach a sufficiently small copy of F_ω so that c is the only common point of $X_1^{(\omega)}$ and of the attached copy. Denote by $X_2^{(\omega)}$ the union of $X_1^{(\omega)}$ and of all attached copies. Thus $X_2^{(\omega)}$ is a dendrite. At the midpoint of each maximal free arc contained in $X_2^{(\omega)}$ we perform the same construction, i.e., we attach a sufficiently small copy of F_ω so that m is the only common point of $X_2^{(\omega)}$ and of the attached copy. Denote by $X_3^{(\omega)}$ the union of $X_2^{(\omega)}$ and of all attached copies. Thus $X_3^{(\omega)}$ is a dendrite. Continuing in this way we obtain an increasing sequence of dendrites

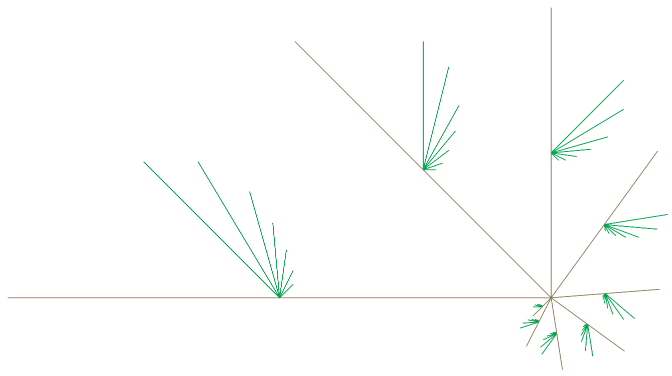


Figure 1.3.6: (A) Ważewski universal dendrite D_ω

$X_1^{(\omega)} \subset X_2^{(\omega)} \subset \cdots \subset X_n^{(\omega)} \subset X_{n+1}^{(\omega)} \subset \ldots$. The construction can be done in the plane in such a way that the limit continuum D_ω defined by

$$D_\omega = \mathrm{cl}\,(\bigcup\{X_n^{(\omega)} : n \in \mathbb{N}\})$$

is again a dendrite. See Figure A.

For another construction of D_ω (using inverse limits) see [Nadler 1992, 10.37, p. 181-185].

The following properties of D_ω are known.

1. D_ω is universal in the class of all dendrites (see e.g. [Nadler 1992, 10.37, p. 181-185]).

2. D_ω is embeddable in the plane (in fact, it is constructed in the plane).

3. Each open image of D_ω is homeomorphic to D_ω (see [Chaaratonik 1980, Theorem 1, p. 490]).

4. D_ω is homogeneous with respect to monotone mappings, [Charatonik 1991, Theorem 7.1, p. 186].

For other mapping properties of D_ω, in particular ones related to the action of the group of autohomeomorphisms on D_ω, see [Charatonik 1995].

1.3.7 Universal dendrites of order m

Let $m \in \{3, 4, \ldots, \omega\}$. By the *standard universal dendrite of order* m we mean a *dendrite* D_m such that each ramification point of D_m is of order m and for each arc $A \subset D_m$ the set of ramification points of D_m which belong to A is dense in A. Their constructions, known again from [Ważewski 1923, Chapter K, p. 137] (see also [Charatonik 1991, (4), p. 168]; for the inverse limit construction see [Chaaratonik 1980, p. 491]), mimic that of the Ważewski universal dendrite

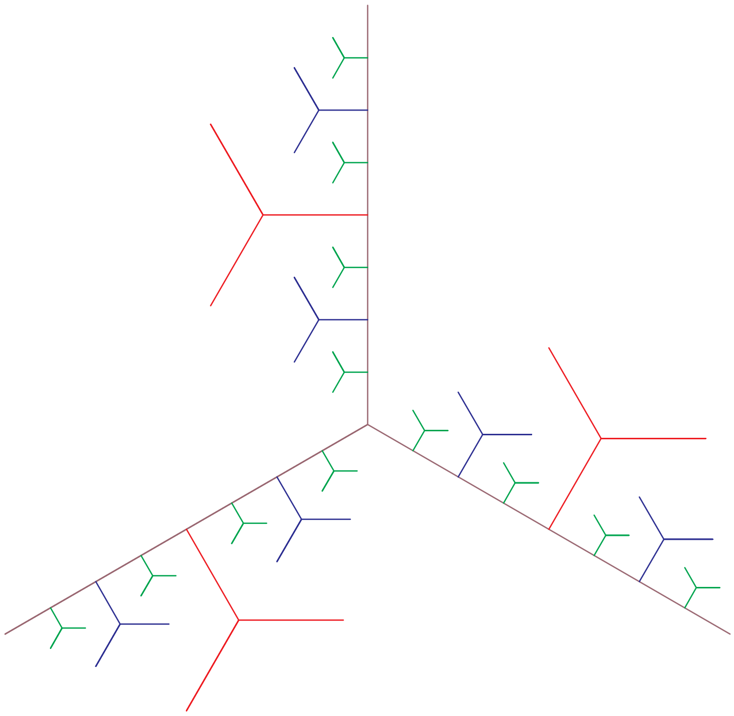


Figure 1.3.7: (A) standard universal dendrite D_3

D_ω, but instead of copies of F_ω we use copies of m-ods at each step of the construction. See Figures A–C for the standard universal dendrites D_3, D_4 and D_6.

The standard universal dendrites D_m have the following properties.

1. For each $m \in \{3, 4, \ldots, \omega\}$ D_m is universal in the class of all dendrites for which the order of their ramification points is less than or equal to m, see e.g. [Menger 1932, Chapter 10, §6, p. 322].

2. If $m, n \in \mathbb{N}$ with $3 \leq m < n$, then there exists an open mapping of D_n onto D_m, [Chaaratonik 1980, Theorem 2, p. 492].

3. Among all standard universal dendrites D_m only D_3 and D_ω are homeomorphic with all their open images, [Chaaratonik 1980, Corollary, p. 493].

4. For each $m \in \{3, 4, \ldots, \omega\}$ a monotone surjection of D_m onto itself is a near homeomorphism if and only if $m = 3$, [Charatonik 1991, Corollary 5.5, p. 178].

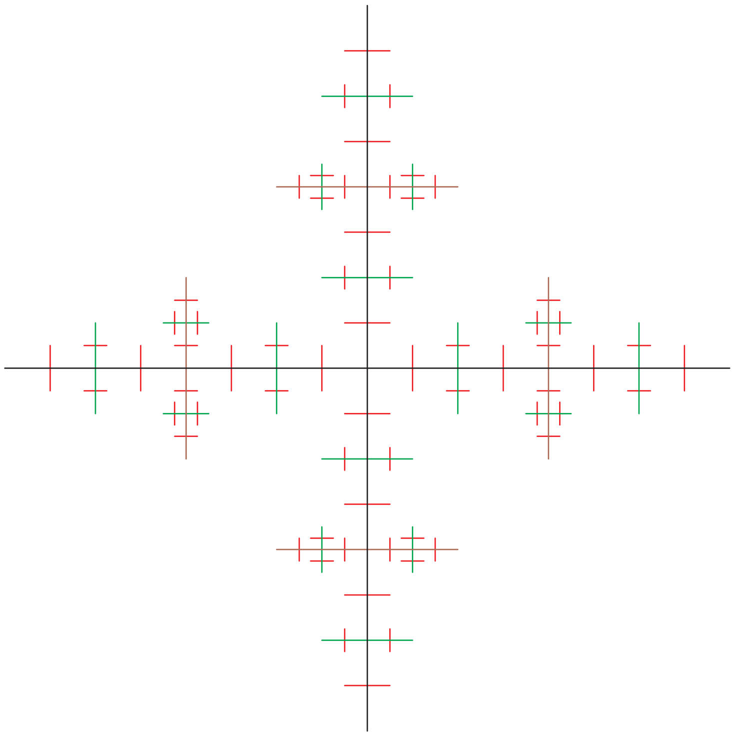


Figure 1.3.7: (B) standard universal dendrite D_4

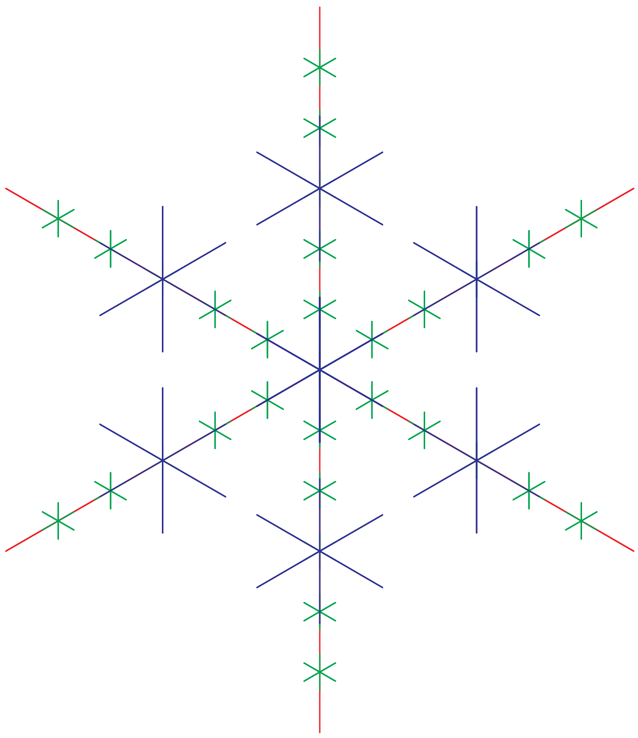


Figure 1.3.7: (C) standard universal dendrite D_6

5. Any two standard universal dendrites D_m and D_n of some orders $m, n \in \{3, 4, \ldots, \omega\}$ are monotonely equivalent, [Charatonik 1991, Corollary 6.6, p. 180].

6. For each $m \in \{3, 4, \ldots, \omega\}$ the dendrite D_m is monotonely homogeneous [Charatonik 1991, Theorem 7.1, p. 186].

Other mapping properties of the standard universal dendrites D_m can be found e.g. in [Chaaratonik 1980], [Charatonik 1991], [Charatonik 1995], [Charatonik et al. 1997a], [Charatonik et al. 1998], [Charatonik et al. 1994] and [Charatonik W.J. et al. 1994].

1.3.8 Other universal dendrites

For a given nonempty set $S \subset \{3, 4, \ldots, \omega\}$ we denote by D_S any dendrite X satisfying the following two conditions:

(a) if p is a ramification point of X, then $\operatorname{ord}(p, X) \in S$;

(b) for each arc A contained in X and for every $m \in S$ there is in A a point p with $\operatorname{ord}(p, X) = m$.

It is shown in [Charatonik W.J. et al. 1994, Theorem 6.2, p. 229] that the *dendrite* D_S is topologically unique, i.e., if two dendrites satisfy conditions (a) and (b) with the same set $S \subset \{3, 4, \ldots, \omega\}$, then they are homeomorphic. The dendrite D_S is called the *standard universal dendrite of orders in* S . If S is a singleton $\{m\}$, then D_S is just the standard universal dendrite D_m defined previously.

The following properties of dendrites D_S are known (see [Charatonik W.J. et al. 1994, Theorems 6.4 and 6.6-6.8, p. 230; Corollary 6.10, p. 232]).

1. For any nonempty subset $S \subset \{3, 4, \ldots, \omega\}$, the dendrite D_S is strongly pointwise self-homeomorphic.

2. If $\omega \in S$, then the dendrite D_S is universal for the family of all dendrites.

3. If the set S is finite with $\max S = m$, then D_S is universal for the family of all dendrites having orders of ramification points at most m.

4. If the set S is infinite and $\omega \notin S$, then D_S is universal for the family of all dendrites having finite orders of ramification points.

5. Nonconstant open images of standard universal dendrites D_S are homeomorphic to D_S if and only if S is a nonempty subset of $\{3, \omega\}$.

6. For any nonempty subset $S \subset \{3, 4, \ldots, \omega\}$ and for an arbitrary dendrite Y there exists a monotone mapping from D_S onto Y, [Charatonik et al. 1998, Theorem 2.22, p. 239].

7. For any nonempty subset $S \subset \{3, 4, \ldots, \omega\}$ the dendrite D_S is monotonely homogeneous, [Charatonik 1996, Theorem 3.3, p. 292].

1.3.9 The dendrite P

In connection with the characterization of dendrites having closed set of their end points (see Property 2 in 1.3.4 above) the classes of those dendrites that do not contain a copy of F_ω and those that do not contain a copy of the comb W are of some interest (see [Arévalo et al. 2001, p. 8]). The first one is just the class of dendrites with finite orders of ramification points, and it is known to have a universal element according to Property 4 in 1.3.8. The other class has also a universal element P. Its construction, given in [Arévalo et al. 2001, p. 9] is the following.

Let p be any point in the plane $\mathbb{R}^2$ and let $p_1, p_2, \ldots$ be a sequence of points in $\mathbb{R}^2$ tending to p and such that no three of the points $p, p_1, p_2, \ldots$ are collinear. Define $P_1 = \bigcup\{\overline{pp_n} : n \in \mathbb{N}\}$. Then P_1 is homeomorphic to F_ω. For every $n, m \in \mathbb{N}$, let p_{nm} be a point in $\mathbb{R}^2 \setminus P_1$ such that for any fixed n the sequence $p_{n1}, p_{n2}, \ldots$ tends to p_n, no three of the points $p_n, p_{n1}, p_{n2}, \ldots$ are collinear, and that the continuum $P_2 = P_1 \cup \bigcup\{\overline{p_n p_{nm}} : n, m \in \mathbb{N}\}$ is a dendrite. We continue constructing dendrites $P_3, P_4, \ldots$ in the same manner, such that we get an increasing sequence of dendrites $P_1 \subset P_2 \subset P_3 \subset \cdots \subset P_n \subset P_{n+1} \subset \ldots$ in $\mathbb{R}^2$ having the property that the closure of their union is also a dendrite. Finally define

$$P = \operatorname{cl}\left(\bigcup\{P_n : n \in \mathbb{N}\}\right).$$

The following properties of the *dendrite* P are proved in [Arévalo et al. 2001, Theorems 4.5 and 4.6, p. 10].

1. A dendrite X is homeomorphic to the dendrite P if and only if each of the following conditions is satisfied:
   (a) X contains no copy of W;
   (b) the set of all end points of X is contained in the closure of the set of all ramification points of X;
   (c) all ramification points of X are of order ω.
2. The dendrite P is universal in the class of all dendrites containing no copy of W.

See Figure A.

1.3.10 Self-homeomorphic dendrites

Below we present some examples of dendrites that illustrate relations between variants of self-homeomorphic properties.

1. There is a strongly pointwise self-homeomorphic dendrite which is not of the form D_S for some $S \subset \{3, 4, \ldots, \omega\}$, see [Charatonik W.J. et al. 1994, Example 6.11, p. 232, and Fig. 4, p. 233]. Note that, according to Property 1 in 1.3.8 each dendrite D_S is

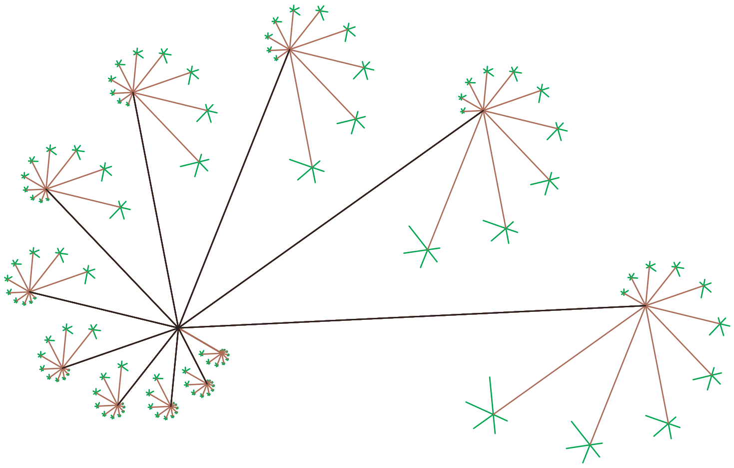


Figure 1.3.9: (A) dendrite P

strongly pointwise self-homeomorphic. This example shows that the opposite implication does not hold. Some additional properties of the example are shown in [Pyrih 1999b, Example 2.1, p. 152]. See Figure A.

2. There is a strongly self-homeomorphic and pointwise self-homeomorphic dendrite which is not strongly pointwise self-homeomorphic, see [Charatonik W.J. et al. 1994, Example 6.12, p. 233].

3. There is a strongly self-homeomorphic dendrite which is not pointwise self-homeomorphic, see [Pyrih 1999a, Example 2.1, p. 572]. Dendrites having this property are studied in [Charatonik et al. XXXXc].

   See Figure B.

1.3.11 Monotone equivalence and monotone homogeneity

As it was mentioned previously, any two universal dendrites D_m and D_n are monotonely equivalent (see Property 5 in 1.3.7). However, the family of all dendrites which are monotonely equivalent to a dendrite D_m contains also other members. To characterize the family we have to construct a special dendrite, denoted L_0.

1.3.12 Omiljanowski dendrite

The *Omiljanowski dendrite* has been constructed by K. Omiljanowski in [Charatonik 1991, Example 6.9, p. 182] (see also [Charatonik 1991, Remark 6.11 and Theorem 6.12, p. 183]). It is defined as the closure of the union of an increasing sequence of dendrites in the plane. We start with the unit straight

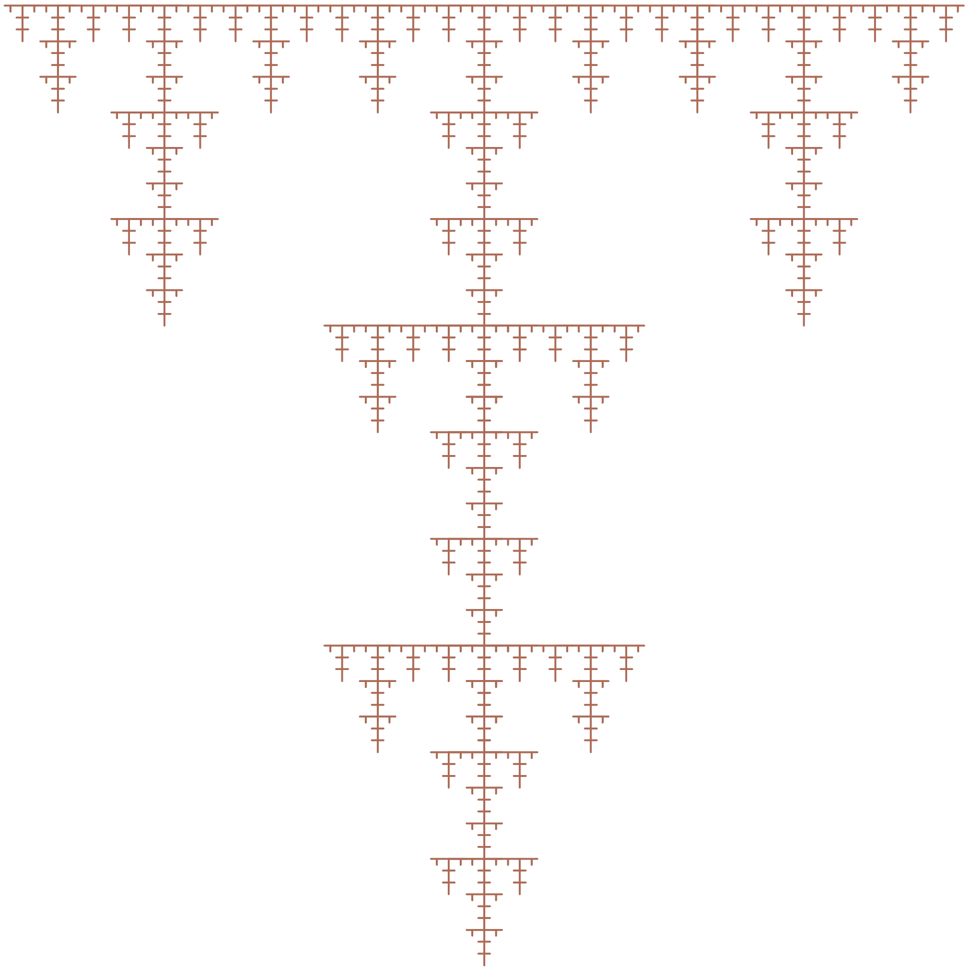


Figure 1.3.10: (A) an example of self-homeomorphic dendrite

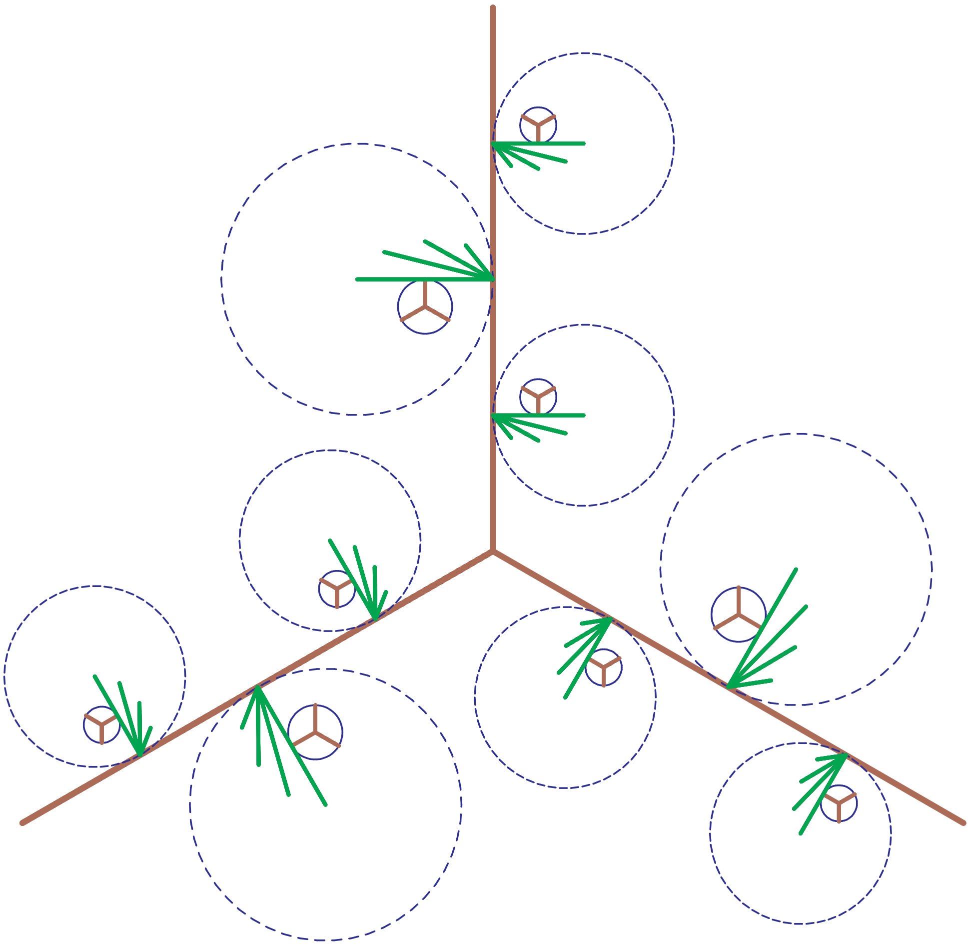


Figure 1.3.10: (B) an example of strongly self-homeomorphic dendrite

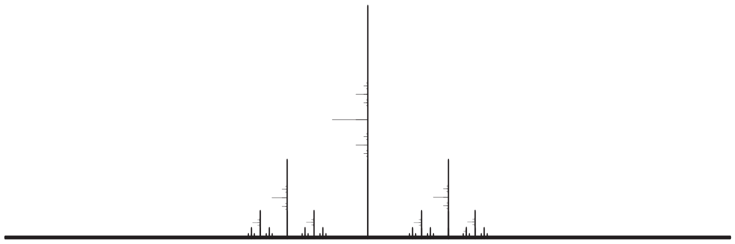


Figure 1.3.12: (A) dendrite L_0

line segment denoted by L_1. Divide L_1 into three equal subsegments and in the middle of them, M, locate a thrice diminished copy of the Cantor ternary set C. At the midpoint of each contiguous interval K to C (i.e., a component K of $M \setminus C$) we erect perpendicularly to L_1 a straight line segment whose length equals length of K. Denote by L_2 the union of L_1 and of all erected segments (there are countably many of them, and their lengths tend to zero). We perform the same construction on each of the added segments: divide such a segment into three equal parts, locate in the middle part M a copy of the Cantor set C properly diminished, at the midpoint of any component K of $M \setminus C$ construct a perpendicular to K segment as long as K is, and denote by L_3 the union of L_2 and of all attached segments. Continuing in this manner we get a sequence of dendrites $L_1 \subset L_2 \subset L_3 \subset \dots$. Finally we put

$$L_0 = \mathrm{cl}\,(\bigcup\{L_i : i \in \mathbb{N}\})$$

and call it a *dendrite* L_0 . See Figure A.

The Omiljanowski dendrite L_0 has the following properties, [Charatonik 1991, Example 6.9, p. 182].

1. All ramification points of L_0 are of order 3.

2. The set $R(L_0)$ of all ramification points of L_0 is discrete (thus nowhere dense).

3. The set of all end points of L_0 is nowhere dense.

4. For each maximal arc A in L_0 the closure of the set $A \cap R(L_0)$ contains a homeomorphic copy of the Cantor set.

5. L_0 is monotonely equivalent to the dendrite D_3.

6. The following conditions are equivalent for a dendrite X (see [Charatonik 1991, Theorem 6.14, p. 185] and [Charatonik et al. 1994, Theorem 5.35, p. 17]; compare [Charatonik et al. 1998, Theorem 2.20, p. 238]):

   (a) X is monotonely equivalent to D_3;

(b) X is monotonely equivalent to D_ω;

(c) X is monotonely equivalent to D_m for each $m \in \{3, 4, \ldots, \omega\}$;

(d) X is monotonely equivalent to D_S for each nonempty set $S \subset \{3, 4, \ldots, \omega\}$;

(e) X is monotonely equivalent to every dendrite Y having dense set $R(Y)$ of its ramification points;

(f) X is monotonely equivalent to some dendrite Y having dense set $R(Y)$ of its ramification points;

(g) X contains a homeomorphic copy of every dendrite L such that its set $R(L)$ of ramification points is discrete and consists of points of order 3 exclusively;

(h) X contains a homeomorphic copy of the dendrite L_0.

According to Property 7 in 1.3.8 (see also Property 6 in 1.3.7) any dendrite D_S is monotonely homogeneous. Another, less restrictive, sufficient condition for monotone homogeneity of a dendrite is the following (see [Charatonik et al. 1997a, Proposition 15, p. 364]).

7. If for a dendrite X the set $R(X)$ of its ramification points is dense in X, then X is monotonely homogeneous.

   The condition $\operatorname{cl} R(X) = X$, being sufficient, is far from being necessary. Namely the Omiljanowski dendrite L_0 has the set $R(L_0)$ discrete, and it is monotonely homogeneous. Moreover, the following statement holds, [Charatonik et al. 1997a, Proposition 20, p. 366].

8. If a dendrite X contains a homeomorphic copy of L_0, then X is monotonely homogeneous.

It would be interesting to know if the converse to the above statement holds true, i.e., if containing the dendrite L_0 characterizes monotonely homogeneous dendrites. In other words, we have the following question.

Question. Does every monotonely homogeneous dendrite contain a homeomorphic copy of L_0?

The above question is closely related to a more general problem.

Problem. Give any structural characterization of monotonely homogeneous.

1.3.13 Chaotic and rigid dendrites

We start presentation of examples related to chaoticity with an example of a dendrite which, while not chaotic, is closely related to the property, and which has been considered as a pattern to construct other examples having various related properties.

1.3.14 Miller dendrite

Recall that a space is said to be incompressible if it is homeomorphic to no proper subspace of itself. The circle S^1 is an example of such a space. In 1925 Zarankiewicz [Zarankiewicz 1925] asked whether there exists an incompressible dendrite. The question has been answered in the positive in 1932 by E. W. Miller [Miller 1932]. The reader is referred to the original Miller's paper for the (rather long) construction of the example. The main idea of the proof is based on the following implication, [Miller 1932, Theorem, p. 831].

If a dendrite X contains a subset K such that

A. each point of K is fixed with respect to any homeomorphism of X onto a subcontinuum of X;

B. each point of X which is not an end point of X lies in an arc contained in X and having its end points in K,

then X is incompressible.

The *Miller dendrite* S has the following properties, [Miller 1932].

1. S contains a subset K having properties A and B above, so it is incompressible.

2. S is not chaotic.

3. S contains open arcs as open subsets, hence it is not rigid.

1.3.15 Dendrites of de Groot-Wille type

An example of a rigid dendrite has been roughly described in [J. de Groot et al. 1958]. Its construction is recalled below (see also [Charatonik 1979, Chapter III, Section 5 , p. 227]).

For any $i \in \mathbb{N}$ let T_i denote the i-od being the union of i straight line unit segments emanating from a point called the origin of T_i. We proceed by induction. Define Y_0 as the unit segment. Let x be the midpoint of Y_0, and define Y_1 as the union of Y_0 and a diminished copy of $T_{j_1} = T_1$ so that the diameter of this copy is less than $1/2$ and that $Y_0 \cap T_1 = \{x\}$. Assume that a tree Y_n is defined such that it is the union of finitely many straight line segments and contains (properly diminished) copies of $T_1, T_2, \ldots, T_{j_n}$, i.e., of the first j_n terms of the sequence $\{T_i\}$. To define Y_{n+1} consider all maximal free segments in Y_n. Note that there are finitely many, say $m(n)$ of them. Let x denote the midpoint of any of these segments. With each point x so defined we associate, in a one-to-one way, a set T_i taken from the $m(n)$ consecutive terms of the sequence $\{T_i\}$, i.e., we use in this step of the construction the sets $T_{j_n+1}, T_{j_n+2}, \ldots, T_{j_{n+1}}$, where $j_{n+1} = j_n + m(n)$. We take each midpoint as the origin of a properly diminished copy of T_i, where $i \in \{j_n + 1, j_n + 2, \ldots, j_{n+1}\}$ in such a way that the diameter of the copy of T_i is less than $1/2^{n+1}$ and that Y_n has only the

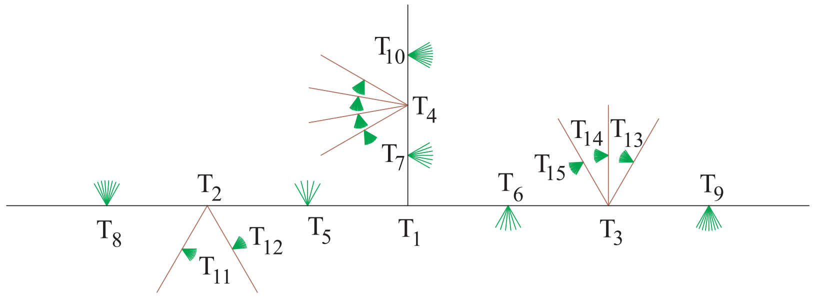


Figure 1.3.15: (A) Y_3

point x in common with the attached copy of T_i. All this can clearly be done so carefully that the resulting set Y_{n+1} is a tree and the limit continuum

$$Y = \operatorname{cl}\left(\bigcup\{Y_n : n \in \mathbb{N}\}\right)$$

is a dendrite. See Figure A.

The following properties of the dendrite Y are shown in [Charatonik 1979, Statement 10, p. 229].

1. The set of all end points of Y is dense in Y.
2. For each $n \in \mathbb{N}$ the set of all points of Y that are of order at least n is dense in Y.
3. For each natural $n \geq 3$ the dendrite Y contains exactly one point of order n.
4. For each point $p \in Y$ and for each component C of $Y \setminus \{p\}$ the continuum $C \cup \{p\}$ contains a homeomorphic copy of Y.
5. Y is not strongly rigid.
6. Y is chaotic.

Recall that for any dendrite conditions 1 and 3 imply 5 and 6, see [Charatonik 1999, Theorem 13, p. 22]. However, neither the above mentioned conditions, nor a rough description given in [J. de Groot et al. 1958], nor the one presented in [Charatonik 1979] lead to a uniquely determined dendrite, because the constructed dendrite Y depends on a function that assigns the consecutive i-ods T_i (used in the succesive steps of the construction) to the midpoints of the maximal free arcs in the trees Y_n. Thus we refer to any of the dendrites Y obtained in this way as to a *dendrite of de Groot-Wille type* .

An example of a dendrite of de Groot-Wille type that is chaotic but not openly chaotic is constructed in [Charatonik 2000, Theorem 3.10, p. 646].

1.3.16 Modified Miller dendrites

The methods used by Miller to construct his example S has also been applied to get a modification D of S in [Charatonik 1979, Chapter IV, p. 230] (an outline of this construction is in [Caratonik 1977]). Below we summarize properties of this example, and recall an extension of this construction.

1. There exists a dendrite D such that:
   (a) each point of D is of order nor greater than 4;
   (b) for each $n \in \{1, 2, 3, 4\}$ the set of all points of D which are of order n is dense in D;
   (c) D is strongly rigid;
   (d) D is chaotic.
2. For any two integers m and n with $3 \leq m < n$ there exists a dendrite $X(m, n)$ such that (see [Charatonik et al. 1996, Theorem 5.5, p. 185] and compare also [Charatonik 1999, Theorem 27, p. 24]):
   (a) $\operatorname{ord}(x, X(m, n)) \in \{1, 2, m, n\}$ for each $x \in X(m, n)$.
   (b) Every arc in $X(m, n)$ contains a point of order m in $X(m, n)$.
   (c) Each of the sets: of all end points of $X(m, n)$, of all points of order m in $X(m, n)$ and of all points of order n in $X(m, n)$ is dense in $X(m, n)$.
   (d) $X(m, n)$ is strongly chaotic.

   Dendrite D is called a *modified Miller dendrite* . (For a further modification of the above construction that leads to chaotic, strongly rigid, openly rigid and not strongly chaotic dendrites, see [Charatonik 2000, Theorem 3.14, p. 650].)

   Finally recall dendrites constructed in [Charatonik 1999, Examples 33 and 35, p. 28].
3. For each natural $n \geq 5$, there exists a strongly rigid and not chaotic dendrite, all points of which are of order at most n ([Charatonik 1999, Example 33, p. 28]).
4. There exists a rigid dendrite which is neither chaotic nor strongly rigid ([Charatonik 1999, Example 35, p. 28]).

1.3.17 Dendrites with the closed set of end points

Dendrites in this class have been already characterized (see Property 2 in 1.3.4). Besides any tree, the locally connected comb W_R comb W_R is an example of a dendrite having the set of its end points closed (and countable). Below we recall a member of the considered class having uncountably many end points.

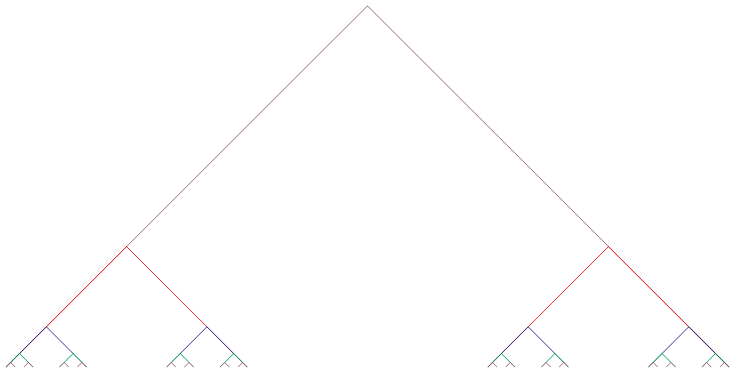


Figure 1.3.18: (A) Gehman dendrite

1.3.18 Gehman dendrite

One of the classical examples of dendrites with its set of end points closed is the *Gehman dendrite* . It can be described as a dendrite G having the set $E(G)$ homeomorphic to the Cantor ternary set C in $[0, 1]$ such that all ramification points of G are of order 3 (see [Gehman 1925, the example on p. 42]; see also [Nikiel 1983, p. 422-423] for a geometrical description; compare [Nikiel 1989, p. 82] and [Nadler 1992, Example 10.39, p. 186]). See Figure A.

The Gehman dendrite has the following properties.

1. The set $R(G)$ of all ramification points of G is discrete.
2. $E(G) = \mathrm{cl}\,(R(G)) \setminus R(G)$.
3. Each dendrite with an uncountable set of its end points contains a homeomorphic copy of the Gehman dendrite, [Arévalo et al. 2001, Proposition 6.8, p. 16].
4. If a continuum contains the Gehman dendrite, then it does not have the periodic-recurrent property, [Charatonik 1998, Theorem 3.3, p. 136].
5. A dendrite X contains the Gehman dendrite if and only if X does not have the periodic-recurrent property, [Illanes 1998, Theorem 2, p. 222].

1.3.19 Modifications of the Gehman dendrite

The concept of the Gehman dendrite has been generalized in [Arévalo et al. 2001, Section 4, p. 4-10] as follows.

1. For each natural number $n \geq 3$ there exists topologically unique dendrite G_n dendrite G_n such that
   (a) $\mathrm{ord}\,(p, G_n) = n$ for each point $p \in R(G_n)$;

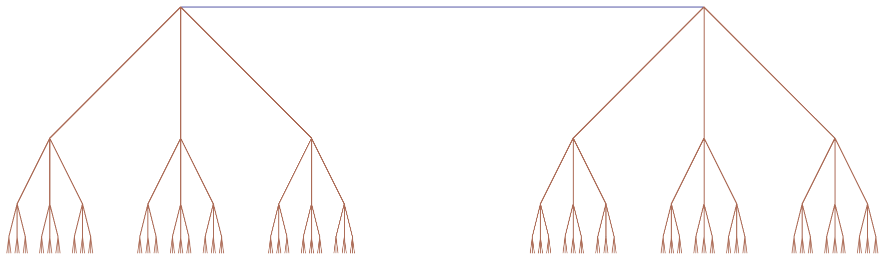


Figure 1.3.19: (A) G_4

(b) $E(G_n)$ is homeomorphic to the Cantor ternary set.

(Note that G_3 is just the Gehman dendrite G.)

2. There exists a *dendrite* G_ω such that

   (a) $\operatorname{ord}(p, G_\omega)$ is finite for each point $p \in G_\omega$;

   (b) $E(G_\omega)$ is homeomorphic to the Cantor ternary set;

   (c) for each natural number n and for each maximal arc A contained in G_ω there is a point $q \in A$ such that $\operatorname{ord}(q, G_\omega) \geq n$.

   See Figure A.

   The dendrites G_n and G_ω have the following universality properties.

3. For each natural number $n \geq 3$ the dendrite G_n is universal for the class of all dendrites X such that $\operatorname{cl} E(X) = E(X)$ and that $\operatorname{ord}(p, X) \leq n$ for each point $p \in X$.

4. Each dendrite G_ω is universal for the class of all dendrites with the closed set of end points.

1.3.20 Mapping hierarchy of dendrites

A classification of topological spaces from the standpoint of the theory of mappings (called also a mapping hierarchy of spaces) is defined in [Borsuk 1959], and it is investigated for dendrites in [Charatonik et al. 1994]. Some questions asked there have already been answered. We recall some examples related to this subject.

1. There exist two dendrites X and Y such that $\{X, Y\}$ has no infimum and no supremum with respect to the class of monotone mappings (see [Charatonik et al. 1994, Example 5.45, p. 18]).

2. There exist dendrites X and Y, both admitting open mappings onto arcs, such that $Y \subset X$ and there is no open mapping from X onto Y (see [Charatonik et al. 1994, Example 6.67, p. 34])

   Indeed, in an arm A of a simple triod Y fix a sequence $\{p_n\}$ of points of A converging to the center of the triod, and take a sequence $\{A_n\}$ of arcs such that $Y \cap A_n = \{p_n\}$ and that $\lim diam_n = 0$. Then $X = Y \cup \bigcup \{A_n : n \in \mathbb{N}\}$ is the needed dendrite.

3. There exists a dendrite which does not admit any open mapping onto an arc and which is the union of two subdendrites admitting open mappings onto an arc (see [Charatonik et al. 1994, Example 6.69, p. 34]).

4. There is a bounded chain of dendrites (ordered with respect to open mappings) that has no supremum (see [Podbrdský et al. 2001]).

5. There is in the plane an uncountable family of dendrites every two members of which are incomparable by r-mappings (see [Sieklucki 1959]).

6. There is a family of dendrites ordered with respect to r-mappings similarly to the segment (see [Sieklucki 1961]).

1.4 Cyclic examples of locally connected continua

(a) Claytor examples of nonplanable curves

(b) Menger compacta, Sierpiński universal plane curve, Menger universal curve; homogeneity of these continua

(c) Other examples

1.4.1 Menger universal continua

The Menger universal continuum M_n^m, for $1 \leq n < m$ [Menger 1926], can be defined as follows [Engelking 1978, pp. 121–122].

Let $\mathcal{F}_0 = \{I^m\}$. Inductively, suppose a collection $\mathcal{F}_k$ of cubes has been defined for $k \geq 0$. Divide each cube $D \in \mathcal{F}_k$ into $3^{m(k+1)}$ congruent cubes with edges of length $\frac{1}{3^{m(k+1)}}$. If $\mathcal{F}_{k+1}(D)$ is the collection of all these cubes that intersect n-dimensional faces of D, then let $\mathcal{F}_{k+1} = \bigcup \{ \mathcal{F}_{k+1}(D) : D \in \mathcal{F}_k \}$.

Define

$$M_n^m = \bigcap_{k=0}^{\infty} \left(\bigcup \mathcal{F}_k \right).$$

The most popular continua are M_1^2, widely known as the Sierpiński universal plane curve or the Sierpiński carpet [Sierpiński 1916], and M_1^3, called the Menger universal curve [Menger 1926]. These particular curves are discussed separately.

Continuum M_n^{2n+1} can also be obtained as the inverse limit $\varprojlim(X_i, p_i^{i+1})$ in the following way. Let $\{G_1, G_2, \dots\}$ be an open base of S^n consisting of open n-cells such that $diam_i \to 0$. Put $X_1 = S^n$ and $X_{i+1} = (X_i \times \{0,1\})/\sim$, where relation $\sim$ identifies points $(x,0),(x,1)$ iff $p_1^i(x) \notin G_i$. The bonding maps p_i^{i+1} are natural retractions [Pasynkov 1965], [Bestvina 1988, pp. 98–99].

1. M_n^m is the universal space in the class of all compacta of dimension $\leq n$ which embed in $\mathbb{R}^m$ [Štanko 1971] ([Menger 1926] for M_{m-1}^m). M_{m-1}^m is universal in the class of all $(m-1)$-dimensional subsets of $\mathbb{R}^m$ ([Engelking 1978, Problem 1.11.D]), and M_n^{2n+1} in the class of all n-dimensional separable metric spaces [Bothe 1963].

2. A continuum X is homeomorphic to M_m^{m+1} if and only if X can be embedded in the $(m+1)$-sphere S^{m+1} in such a way that $S^{m+1} \setminus X$ has infinitely many components $C_1, C_2, \dots$ such that $diam_i \to 0$, $\operatorname{bd} C_i \cap \operatorname{bd} C_j = \emptyset$ for $i \neq j$, $\operatorname{bd} C_i$ is an m-cell for each i and $\bigcup_{i=1}^{\infty} \operatorname{bd} C_i$ is dense in X (see [Cannon 1973] and [Chigogidze et al. 1995, Theorem 6.1.2, p. 74]).

3. A continuum X is homeomorphic to M_n^{2n+1} if and only if X is n-dimensional, C^{n-1} and LC^{n-1}-space with the disjoint n-disks property (DD^nP property) [Bestvina 1988, Corollary 5.2.3, p.98].

   Equivalently, X is homeomorphic to M_n^{2n+1} if and only if X is n-dimensional, C^{n-1} and LC^{n-1}-space such that any continuous map from a compact space of dimension $\leq n$ into M_n^{2n+1} is a uniform limit of embeddings (Z-embeddings) [Bestvina 1988, Theorem 2.3.8, p. 36].

4. (a) If $Z \subset M_n^{2n+1}$ is a Z-setZ-set, then, for every open neighborhood $U \subset M_n^{2n+1}$ of Z and every $\epsilon > 0$, there is $\delta > 0$ such that if $Z' \subset M_n^{2n+1}$ is a Z-set and $h : Z \to Z'$ is a δ-homeomorphism, then h extends to an ϵ-homeomorphism $h^* : M_n^{2n+1} \to M_n^{2n+1}$ such that $h^*|M_n^{2n+1} \setminus U =$ identity [Bestvina 1988, Theorem 3.1.1, p. 65].

   (b) For every $\epsilon > 0$, there exists $\delta > 0$ such that if Z, Z' are Z-setsZ-set in M_n^{2n+1} and $h : Z \to Z'$ is a δ-homeomorphism, then there is an ϵ-homeomorphism $h^* : M_n^{2n+1} \to M_n^{2n+1}$ extending h [Bestvina 1988, Theorem 3.1.3, p. 71].

   (c) Every homeomorphism between Z-sets in M_n^{2n+1} extends to an autohomeomorphism of M_n^{2n+1} [Bestvina 1988, Corollary 3.1.5, p. 72].

   It follows that M_n^{2n+1} is strongly locally homogeneous and countably dense homogeneous.

5. If $m < 2n+1$, then M_n^m is not homogeneous [Lewis 1987] (for $m \geq 2n+1$ the homogeneity of M_n^m follows from the property above).

6. $M_n^{2n+1} \times X$ is not 2-homogeneous for an arbitrary continuum X [Kuperberg et al. 1995].

7. The groups of autohomeomorphisms of M_n^{2n+1} and M_n^{n+1} are Polish and one-dimensional (see [Oversteegen et al. 1994, Corollary 6] and Property 2).

   The group is totally disconnected for M_n^{2n+1} (see [Brechner 1966, Theorem 1.3] for $n=1$ and its natural extension for any n).

8. The Polish group of autohomeomorphisms of M_n^m is at most one-dimensional [Oversteegen et al. 1994].

9. The group of autohomeomorphisms of M_n^{2n+1} is simple (see [Chigogidze et al. 1995, Theorem 3.2.4]).

10. Every autohomeomorphism of M_n^{2n+1} is a composition of two homeomorphisms, each of which is the identity on some nonempty open set [Sakai 1994].

11. For each $t \in [n, 2n+1]$, there is a copy of $M_n^{2n+1} \subset \mathbb{R}^{2n+1}$ with the Hausdorff dimension t [Chigogidze et al. 1995, Theorem 4.2.6].

12. Every Z-set in M_n^{2n+1} is the fixed point set of some autohomeomorphisms of M_n^{2n+1} [Sakai 1997].

13. Any C^{n-1} and LC^{n-1}-continuum is the image of M_n^{2n+1} under a UV^{n-1}-map [Bestvina 1988, Theorem 5.1.8, p. 95].

14. If a Polish space X contains a topological copy M of M_n^{2n+1} or M_n^{n+1}, then the space of all copies of M in X with the Hausdorff metric is not a $G_{\delta\sigma}$-set [Krupski XXXXb].

1.4.2 Sierpiński carpet

The *Sierpiński universal plane curve* or the *Sierpiński carpet* [Sierpiński 1916] M_1^2 is a well known fractal obtained as the set remaining when one begins with the unit square I^2 and applies the operation of dividing it into 9 congruent squares and deleting the interior of the central one, then repeats this operation on each of the surviving 8 squares, and so on. See Figure A.

1. M_1^2 is universal in the class of all at most one-dimensional subsets of the plane (equivalently, of all boundary subsets of the plane) [Sierpiński 1916], [Sierpiński 1922].

2. The following statements are equivalent:

   (a) X is homeomorphic to M_1^2;

   (b) X is a locally connected plane curve that contains no local cut points;

   (c) X is a continuum embeddable in the plane in such a way that $\mathbb{R}^2 \setminus X$ has infinitely many components $C_1, C_2, \ldots$ such that $diam_i \to 0$, $\operatorname{bd} C_i \cap \operatorname{bd} C_j = \emptyset$ for $i \neq j$, $\operatorname{bd} C_i$ is a simple closed curve for each i and $\bigcup_{i=1}^{\infty} \operatorname{bd} C_i$ is dense in X [Whyburn 1958].

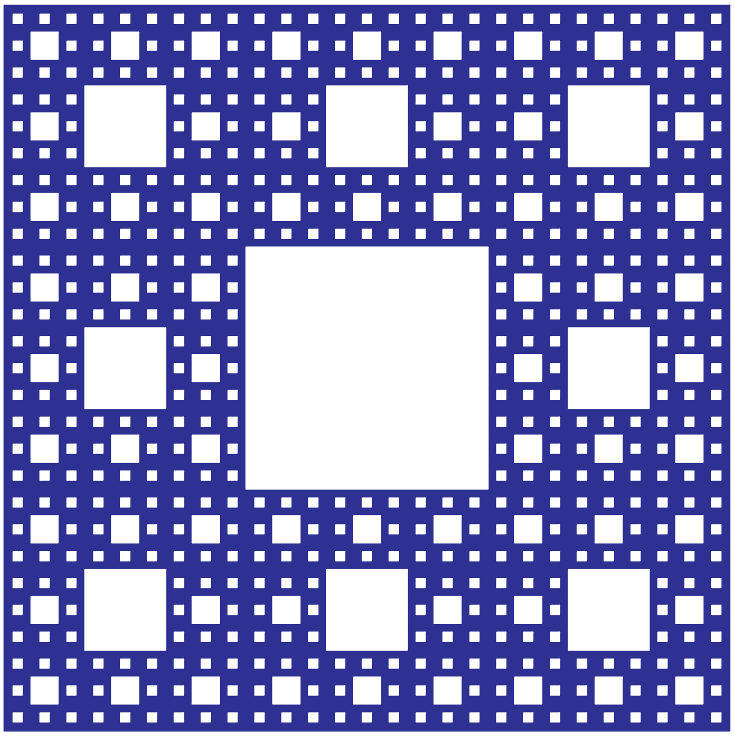


Figure 1.4.2: (A) Sierpiński carpet

3. A complete metric space X contains a topological copy of M_1^2 if and only if X contains a subset with the bypass property [Prajs 1998a].

4. The group of all autohomeomorphisms of M_1^2 has exactly two orbits: one of them is the union of all simple closed curves which are the boundaries of complementary domains of M_1^2 [Krasinkiewicz 1969].

   The group is a Polish topological group which is totally disconnected and one-dimensional (see [Brechner 1966, Theorem 1.2] and Property 7 in 1.4.1).

5. Any homeomorphism between Cartesian products of copies of the Sierpiński carpet is factor preserving [Kennedy Phelps 1980]. Consequently, no such product is homogeneous.

6. The Sierpiński carpet can be continuously decomposed into pseudo-arcs such that the decomposition space is homeomorphic to the carpet [Prajs 1998b, Corollary 18], [Seaquist 1995]. In fact, M_1^2 is the only planar locally connected curve admitting such a decomposition [Prajs 1998b, Corollary 18].

7. The Sierpiński carpet is homogeneous with respect to monotone open mappings [Prajs 1998b, Corollary 24], [Seaquist 1999]. Moreover, every continuum which is locally homeomorphic to M_1^2 (i. e., M_1^2-manifold) is homogeneous with respect to monotone open mappings [Prajs 1998b, Theorem 23].

8. M_1^2 is homogeneous with respect to the class of simple mappings [Charatonik 1984].

9. If C is a curve, then the set of all mappings $f : C \to \mathbb{R}^2$ such that $f(C)$ is homeomorphic to M_1^2 is a residual subset of the space $(\mathbb{R}^2)^C$ of all mappings of C into $\mathbb{R}^2$ with the uniform convergence metric.

   If $C(X)$ is the hyperspace of all subcontinua of a compact space X and $C_1(X)$ its subspace of all curves, then the set

   $$\{\,(C, f) \in C(X) \times (\mathbb{R}^2)^X : f(C) \text{ is homeomorphic to } M_1^2\,\}$$

   is residual in $C(X) \times (\mathbb{R}^2)^X$; in other words, almost all mappings in $(\mathbb{R}^2)^X$ map almost all curves in X onto copies of M_1^2, where "almost all" means all with except of a subset of the first category in corresponding spaces [Mazurkiewicz 1938].

10. If a locally compact space X contains a topological copy of the Sierpiński carpet, then the space of all copies of the Sierpiński carpets in X with the Hausdorff metricHausdorff metric is a true absolute $F_{\sigma\delta}$-set [Krupski XXXXb].

1.4.3 Menger universal curve

The *Menger universal curve* M_1^3 is the subset of the unit cube I^3 whose projections onto faces of the cube are the Sierpiński carpets, i.e.,

$$M_1^3 = \{ (x,y,z) \in I^3 : (x,y) \in M_1^2, \quad (y,z) \in M_1^2, \quad (x,z) \in M_1^2 \}$$

[Menger 1926]. Other descriptions can be found in 1.4.1 (see also [Mayer et al. 1986, pp. 5–6, 8–9]). See Figure A.

1. The following statements are equivalent:

   (a) X is homeomorphic to M_1^3;

   (b) X is a locally connected curve with no local cut points and no planar open nonempty subsets [Anderson 1958, Theorem XII, p. 13];

   (c) X is a homogeneous locally connected curve different from a simple close curve [Anderson 1958, Theorem XIII, p. 14];

   (d) X is a locally connected curve with the disjoint arcs property (see Property 3 in 1.4.1);

   (e) X is a locally connected curve and each arc in X is approximately non-locally-separating arc and has empty interior in X [Krupski et al. XXXXa, Theorem 3, p. 86].

2. M_1^3 is universal in the class of all metric separable spaces of dimension ≤ 1 [Menger 1926].

3. Z-setsZ-set in M_1^3 coincide with non-locally-separating closed subsets of M_1^3.

   If Z is a separable metric space of dimension ≤ 1, then Z can be embedded as a non-locally-separating subset of M_1^3 [Mayer et al. 1986, Theorem 6.1, p. 42].

4. If Z is a closed subset of a metric space X, $\dim(X \setminus Z) \leq 1$ and $f : Z \to M_1^3$ is a continuous mapping with non-locally separating image $f(Z)$, then f can be extended to a map $g : X \to M_1^3$ such that $g|X \setminus Z$ is an embedding into $M_1^3 \setminus f(Z)$ [Mayer et al. 1986, Theorem 6.4, p. 44].

5. Every continuous surjection h between non-locally separating closed subsets Z, Z' of M_1^3 extends to a mapping h^* of M_1^3 such that $h^*|M_1^3 \setminus Z$ is a homeomorphism onto $M_1^3 \setminus Z'$ [Mayer et al. 1986, Corollary 6.5, p. 44]; moreover, for every $\epsilon > 0$ there exists $\delta > 0$ such that if h is a δ-homeomorphism, then h^* can be taken as an ϵ-homeomorphism (see Property 4b in 1.4.1).

6. For each locally connected continuum X, there exist open surjections $f : M_1^3 \to X$ and $g : M_1^3 \to X$ such that $f^{-1}(x)$ is homeomorphic to M_1^3 and $g^{-1}(x)$ is a Cantor set, for any $x \in X$ [Wilson 1972].

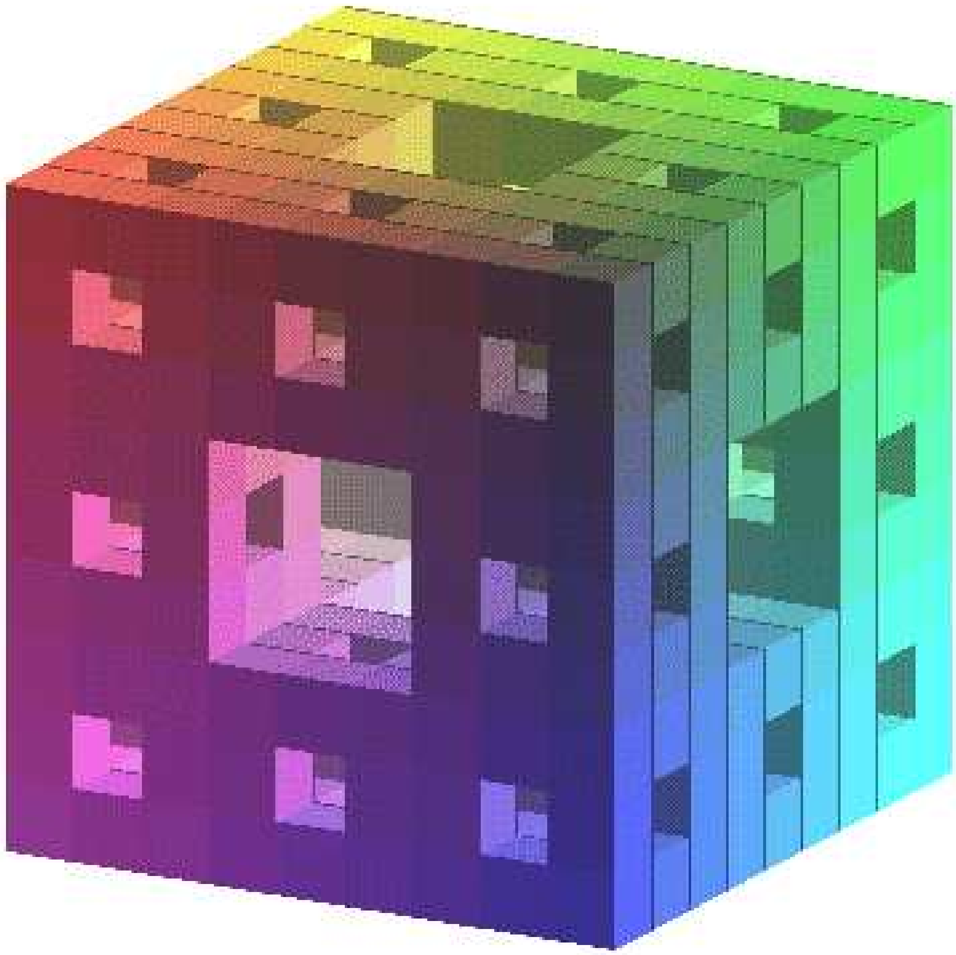


Figure 1.4.3: (A) Menger universal curve

7. Any compact 0-dimensional group G acts freely on M_1^3 so that the orbit space M_1^3/G is homeomorphic to M_1^3 [Anderson 1957, Theorem 1].
8. If a locally compact space X contains a topological copy of M_1^3, then the space of all copies of M_1^3 in X with the Hausdorff metricHausdorff metric is a true absolute $F_{\sigma\delta}$-set [Krupski XXXXb].

1.5 2-dimensional locally connected continua

(a) Surfaces, polyhedra

(b) Pontryagin surfaces, Pontryagin disk (sphere)

(c) 2-dimensional AR containing no disk

(d) Borsuk's hat

(e) Alexander horned sphere

(f) Bing dogbone space

(g) Other examples

1.6 Higher dimensional locally connected continua

(a) n-dimensional manifolds

(b) Hilbert cube manifolds

(c) Other important examples

1.7 Variations of local connectedness

(a) Hereditary local connectedness

(b) Connectedness im kleinen

(c) Semi-local connectedness

Chapter 2

JOINING PROPERTIES

2.1 Cut points and separating points (also local)

2.2 Arcwise connectedness; arc components

2.3 Continuum chainability

2.4 Various kinds of connectedness (λ-connectedness, δ-connectedness, etc.)

Chapter 3

DECOMPOSABLE AND INDECOMPOSABLE CONTINUA

3.1 Decomposable continua (also hereditary)

3.2 Indecomposable continua (also hereditary)

(a) Lakes of Wada

(b) Knaster type continua; Brouwer-Janiszewski-Knaster (buckethandle) continuum

(c) Other simple examples

(d) The pseudo-arc

3.2.1 Knaster continua

By a *Knaster continuum* is meant a continuum homeomorphic to the inverse limit $\varprojlim(I_n, f_n)$ of a sequence of unit intervals $I_n = I$ with open, non-homeomorphic bonding maps f_n. Denote by $\mathcal{K}$ the class of all Knaster continua.

The most popular examples of Knaster continua is the *buckethandle* (sometimes called by dynamists the *horseshoe*) B_1, where $f_n(t) = 1 - |2t - 1|$, and the *double buckethandle* B_2 with

$$f_n(t) = \begin{cases} 3t & \text{if } 0 \le t \le \frac{1}{3}, \\ 2 - 3t & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ 3t - 2 & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

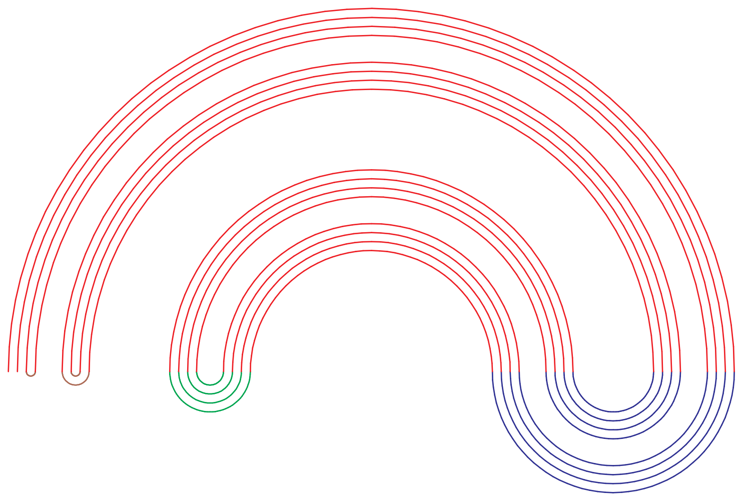


Figure 3.2.1: (A) buckethandle

Nice geometric models of B_1 and B_2 can be described in the following way (see [Kuratowski 1968, pp. 204–205]). If C denotes the standard Cantor ternary set in I, then B_1 is homeomorphic to the union of all semi-circles

$$\{ (x, y) \in \mathbb{R}^2 : (x - \frac{1}{2})^2 + y^2 = r^2, y \geq 0 \},$$

where $r + \frac{1}{2} \in C$, and all semi-circles

$$\{ (x, y) : (x - \frac{5}{2 \cdot 3^n})^2 + y^2 = r^2, y \leq 0 \},$$

for $n \in \mathbb{N}$, where $r + \frac{5}{2 \cdot 3^n} \in C \cap [\frac{2}{3^n}, \frac{1}{3^{n-1}}]$.

See Figure A.

If E is the quintary Cantor set in I (real numbers in I which can be written in the enumeration system at base 5 without digits 1 and 3), then B_2 is homeomorphic to the union of all semi-circles

$$\{ (x, y) : (x - \frac{7}{10 \cdot 5^n})^2 + y^2 = r^2, y \leq 0 \},$$

for $n \in \mathbb{N}$ and $r + \frac{7}{10 \cdot 5^n} \in E \cap [\frac{2}{5^{n+1}}, \frac{1}{5^n}]$ and all semi-circles

$$\{ (x, y) : (x - (1 - \frac{7}{10 \cdot 5^n}))^2 + y^2 = r^2, y \geq 0 \},$$

for $n \geq 0$ and $\frac{2}{5^{n+1}} \leq \frac{7}{10 \cdot 5^n} - r \leq \frac{1}{5^n}$.

See Figure B.

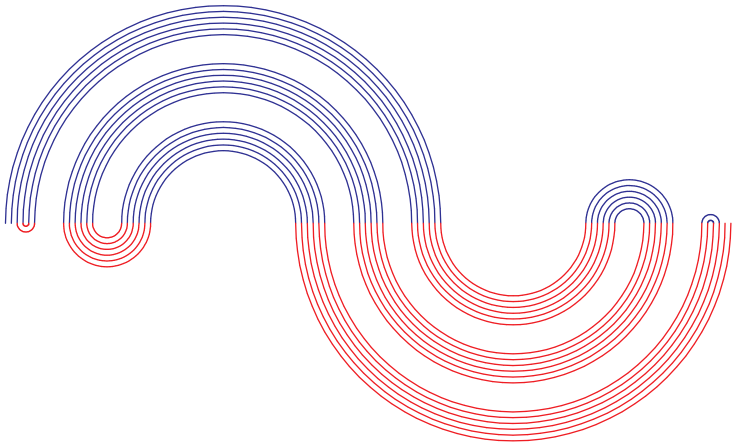


Figure 3.2.1: (B) double buckethandle

1. If $w_p : I \to I$ is the map such that $w_p(i/p) = 0$ for even i, $w_k(i/p) = 1$ for odd i, where i is an integer between 0 and $p \in \mathbb{N}$, and w_p is linear on each interval $[i/p, (i+1)/p]$, then $\mathcal{K}$ is equal to the family of all continua homeomorphic to inverse limits $\varprojlim(I_n, f_n)$, where, for each n, $I_n = I$ and $f_n = w_{p_n}$ for some p_n [Rogers 1970]. So, each $K \in \mathcal{K}$ is determined by a sequence $\mathbf{p} = (p_1, p_2, \dots)$ such that $K = \varprojlim(I_n, w_{p_n})$. We will denote such K by $K(\mathbf{p})$ and call it a **p**-*adic Knaster continuum* . Without loss of generality, one can assume that all p_n's are prime.

2. Each **p**-adic Knaster continuum K is homeomorphic to the quotient of the **p**-adic solenoid $\Sigma(\mathbf{p})$ under the relation

$$a \sim b \Leftrightarrow a = b \quad \text{or} \quad a = b^{-1},$$

where $a, b \in \Sigma(\mathbf{p})$ and b^{-1} is the group inverse of b. A point of K is an end point of K if and only if it is an image under the quotient map of the neutral element or the element of order 2 (if such exists) of the corresponding solenoid (see, e.g., [Krupski 1984b, p. 43]).

3. The class $\mathcal{K}$ of all Knaster continua is equal to the class of all arc-like continua with one or two end points having the property of Kelley and arcs as proper non-degenerate subcontinua and which themselves are not arcs [Krupski 1984b]. In particular, all of them are indecomposable.

4. There are $2^{\aleph_0}$ mutually non-homeomorphic members of $\mathcal{K}$ [Wa]; in fact, there is a subfamily of $\mathcal{K}$ of cardinality $2^{\aleph_0}$ no member of which is an open image of another one [Dębski 1985].

5. Any open map between two Knaster continua $\varprojlim(I_n, f_n)$, $\varprojlim(I_n, g_n)$ is the uniform limit of open maps which are induced by maps $(I_n, f_n) \to$

(I_n, g_n) between the inverse sequences [Eberhart et al. 1999, Theorem 4.8, p. 145].

6. If $K \in \mathcal{K}$, then every indecomposable continuum can be mapped onto K [Rogers 1970, Corollary, p. 455].

   Moreover, if X is an indecomposable (Hausdorff) continuum, $a, b \in X$ and $a \neq b$ (a and b belong to different composants of X), then there is a continuous surjection $f : X \to B_1$ such that $f(a) \neq f(b)$ ($f(a)$ and $f(b)$ belong to different composants of B_1, resp.). It follows that any indecomposable continuum embeds in the Cartesian product of copies of B_1 [Bellamy 1973, Corollaries 1 and 2, p. 305].

7. If $K_1, K_2 \in \mathcal{K}$ then there are $2^{\aleph_0}$ distinct homotopy types of maps of K_1 onto K_2 that map an end point of K_1 onto an end point of K_2 [Minc 1999].

8. If $K \in \mathcal{K}$ and f is a monotone map of K, then $f(K)$ is homeomorphic to K [Krupski 1984b].

9. There is no exactly 2-to-1 map defined on a Knaster continuum [Dębski 1992].

10. Knaster continua are absolute retracts for the class of all hereditarily unicoherent continua, i.e., if $K \in \mathcal{K}$, X is a hereditarily unicoherent continuum and $K \subset X$, then K is a retract of X. Equivalently, if A is a closed subset of a hereditarily unicoherent continuum X, then every map f from A onto a Knaster continuum can be extended over X ([Maćkowiak 1984] for $K = B_1$ and X a Hausdorff hereditarily unicoherent continuum and [Charatonik et al. XXXXa] for arbitrary $K \in \mathcal{K}$).

11. Any two points of a plane continuum X can be joined by a hereditarily decomposable subcontinuum if and only if X cannot be mapped onto B_1 [Hagopian 1974, Theorem 2, p. 133].

12. Every autohomeomorphism of the buckethandle B_1 is isotopic to some iterate of the shift homeomorphism $f : B_1 \to B_1, f(x_1, x_2, \dots) = (x_2, x_3, \dots)$ [Aarts et al. 1991, Theorem 4.4, p. 204].

    Moreover, any autohomeomorphism f of a Knaster continuum $\varprojlim(I_n, w_{p_n})$ is isotopic to a standard homeomorphism g (defined as a map induced by a map $(I_n, w_{p_n}) \to (I_n, w_{p_n})$ of inverse systems); maps f and g have the same topological entropy and if the entropy is positive, then they are semi-conjugate [Kwapisz 2001, Theorem 4, p. 271].

13. If e is the end point of B_1 and S, T are composants of B_1 which do not contain e, then there exists a continuous injection $h : B_1 \setminus \{e\} \to B_1 \setminus \{e\}$ such that $h(S) = T$ [Aarts et al. 1991, Theorem 5.1].

    Any two composants of B_1 which do not contain e are homeomorphic [Bandt 1994].

14. Every autohomeomorphism of the buckethandle continuum has at least two fixed points [Aarts et al. 1998].

15. If A is a Borel subset of $K \in \mathcal{K}$ and A is the union of a family of composants of K, then A is meager or comeager [Emeryk 1980], [Krasinkiewicz 1974a].

16. Each composant C of B_1 different from the one containing the end point is internal, i.e., every continuum $L \subset \mathbb{R}^2$ intersecting both C and $\mathbb{R}^2 \setminus B_1$ must intersect all composants of B_1 [Krasinkiewicz 1974b, Theorem, p. 261].

17. Any Knaster continuum $K = \varprojlim(I_n, f_n)$, where $f_m = f_n$ for each m, n, is homeomorphic to the attracting set of a homeomorphism $h : D^2 \to h(D^2) \subset D^2$ of the plane closed disk [Barge 1986]. If $f_n(t) = 1 - |2t - 1|$ for all n (i.e., $K = B_1$), then h can be defined as a diffeomorphism which is called a *horse-shoe map*—it was first described by S. Smale in [Smale 1965] (see also [Barge 1988]). See Figure C.

18. The buckethandle B_1 and all Knaster continua with two end points admit no mean [Illanes XXXX], [Kawamura et al. 1996, Theorem 2.2, p. 99].

19. The hyperspace $C(K)$ of all subcontinua of any Knaster continuum K is homeomorphic to the cone $Cone(K)$ over K by a homeomorphism $h : C(K) \to Cone(K)$ which sends K onto the vertex of $Cone(K)$ and the set of all singletons of K onto the base of the cone [Dilks et al. 1981, Corollary 12, p. 639].

3.3 Unicoherent continua (also hereditary)

3.4 Discoherent continua

3.5 Multicoherent continua

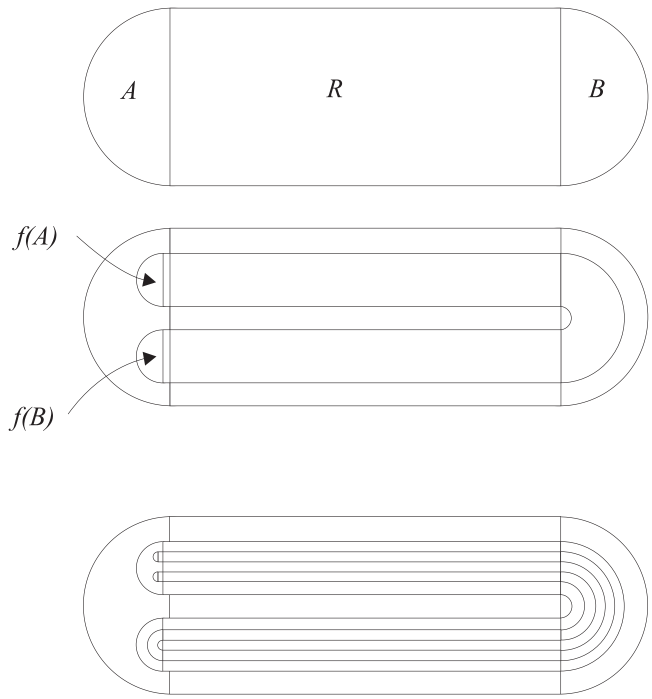


Figure 3.2.1: (C) the horse-shoe map f maps the stadium shaped region D $(A \cup R \cup B)$ in such a way, that it shrinks R vertically, stretches R horizontally, contracts the semicircles then folds the space once and places this result into itself so that $f(A)$ and $f(B)$ are in the interior of A and $f(R)$ is in the interior of D.

Chapter 4

IRREDUCIBLE CONTINUA

4.1 Elementary examples

(a) An arc, Sin Curve

(b) Other examples (Cantor Organ, Cantor Accordion, etc.)

4.1.1 Sin curve

The *(topologist's) sin curve* S is defined by

$$S = \{ (x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, 0 < x \leq 1 \} \cup \{ (0, y) : -1 \leq y \leq 1 \}.$$

1. S is irreducible between points $(0, y)$ and $(1, \sin 1)$, $-1 \leq y \leq 1$. It has exactly three end points and two arc components.

2. It is one of the simplest arc-like continua.

3. It is a compactification of a ray $(0, 1]$ with remainder an arc.

4. It has the periodic-recurrent property [Charatonik et al. 1997b, Corollary 5.10, p. 117].

5. The only possible confluent nondegenerate images of S are an arc and a continuum homeomorphic to S [Nadler 1977a].

6. The hyperspace $C(S)$ of all subcontinua of S is homeomorphic to the cone over S [Nadler 1977b].

See Figure A.

There are many variations of the sin curve. Some of them are pictured below. See Figure B–C.

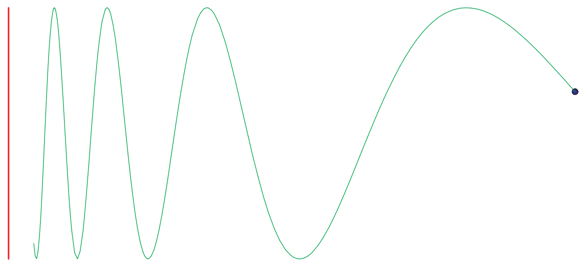


Figure 4.1.1: (A) sin curve

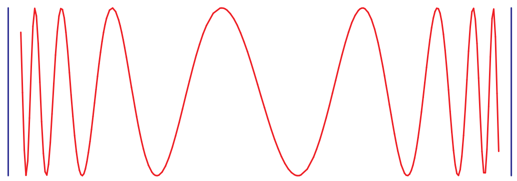


Figure 4.1.1: (B) union of two sin curves

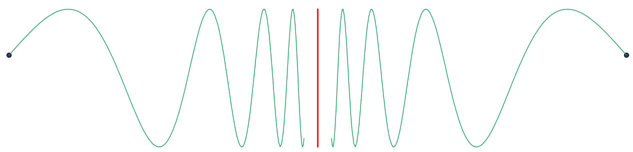


Figure 4.1.1: (C) union of two sin curves

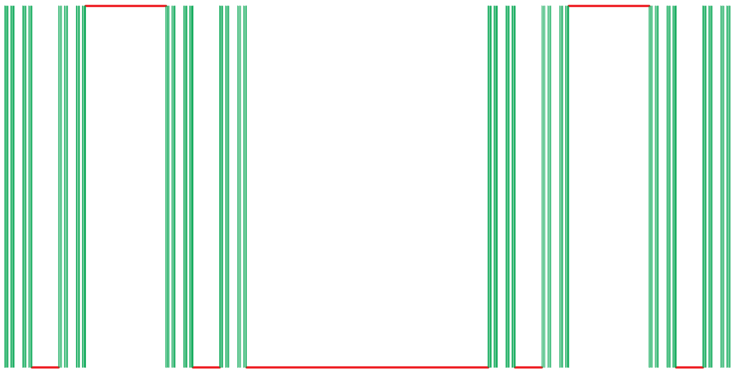


Figure 4.1.2: (A) Cantor organ

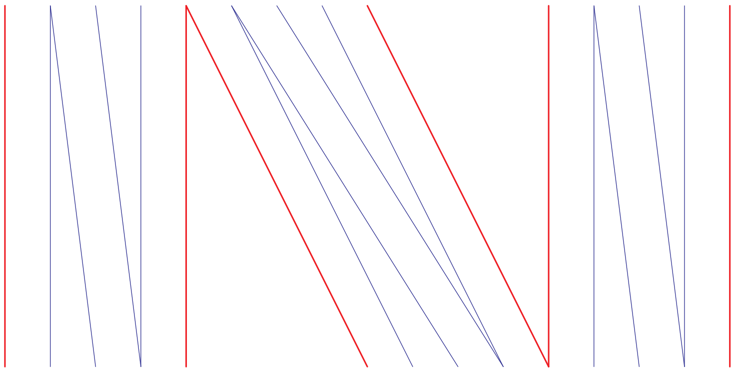


Figure 4.1.2: (B) Cantor accordion

4.1.2 Cantor organ and accordion

The *Cantor organ* X is the union of the product $C \times I$ of the Cantor ternary set C and the unit interval I and all segments of the form $J \times \{0\}$ or $J' \times \{1\}$, where J (J', resp.) is the closure of a component of $I \setminus C$ of length $1/3^{2k-1}$ $(1/3^{2k})$, $k \in \mathbb{N}$ [Kuratowski 1968, p. 191]. See Figure A.

1. X is an arc-like continuum which is irreducible between points $(0, x)$ and $(1, y)$, where $0 \leq x, y \leq 1$, and has exactly four end points.

2. It has uncountably many arc components.

A variation of the Cantor organ is the *Cantor accordion* Y which is defined as the monotone image of X under a map that shrinks horizontal bars $J \times \{0\}$ and $J' \times \{1\}$ to points [Kuratowski 1968, p. 191]. See Figure B.

Besides the above properties,

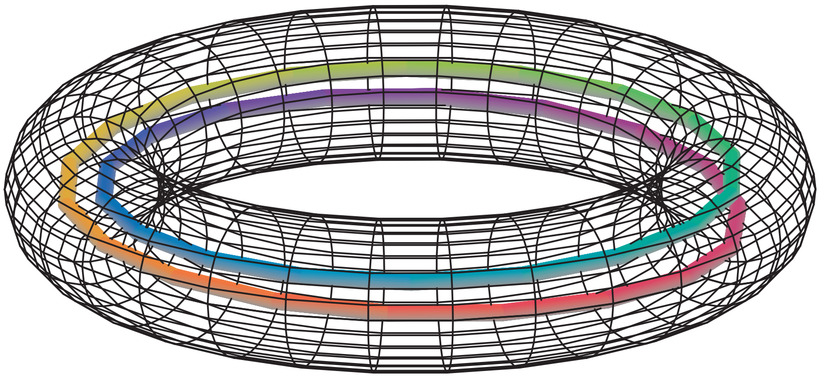


Figure 4.3.1: (A) dyadic selenoid

- Y has an upper semi-continuous monotone decomposition into arcs with the quotient space an arc.

4.2 Arc-like (chainable) continua

(a) The simplest arclike continua with n end points

(b) The pseudo-arc

4.3 Irreducible circle-like continua

(a) Solenoids

(b) The pseudo-circle, pseudo-solenoids

4.3.1 Solenoids

A *solenoid* is a continuum homeomorphic to the inverse limit $\Sigma(\mathbf{p}) = \varprojlim(S^1, f_n)$ of the inverse sequence of unit circles S^1 in the complex plane with bonding maps $f_n(z) = z^{p_n}$, where $\mathbf{p} = (p_1, p_2, \dots)$ is a sequence of prime numbers; it is called a $\mathbf{p}$-*adic solenoid* . The solenoid $\Sigma(2, 2, \dots)$ is known as a *dyadic solenoid* .

Geometrically, solenoid $\Sigma(\mathbf{p})$ can be described as the intersection of a sequence of solid tori $T_1 \supset T_2 \supset \dots$ such that T_{n+1} wraps p_n times around T_n without folding and T_n is ϵ_n-thin, for each $n \in \mathbb{N}$, where $\lim \epsilon_n = 0$. See Figure A.

1. Each solenoid can be constructed (up to a homeomorphism) as the quotient space of the product $C \times I$ by identifying each point $(c, 1)$ with $(h(c), 0)$, where $h : C \to C$ is a homeomorphism of the Cantor set C such

that for every $\epsilon > 0$ there exist a closed-open subset D of C and a positive integer n such that $\{f^j(D)\colon j = 1, \cdots, n\}$ is a cover of C consisting of pairwise disjoint subsets of C with diameters less than ϵ [Gutek 1980].

2. Each solenoid $\Sigma(\mathbf{p})$ is an Abelian topological group with a group operation $(z_1, z_2, \dots) \cdot (z_1', z_2', \dots) = (z_1 \cdot z_1', z_2 \cdot z_2', \dots)$ and the neutral element $e = (1, 1, \dots)$.

3. Either of the following conditions is equivalent for a nondegenerate continuum X different from a simple closed curve to be a solenoid.

   (a) X is homeomorhic to a one-dimensional topological group [Hewitt 1963];

   (b) X is indecomposable and is homeomorphic to a topological group [Chigogidze 1996, Theorem 8.6.18];

   (c) X is circle-like, has the property of Kelley and contains no local end point [Krupski 1984c, Theorem (4.3)];

   (d) X is circle-like, has the property of Kelley, each proper nondegenerate subcontinuum of X is an arc and X has no end pointsend point;

   (e) X is circle-like, has the property of Kelley and has an open cover by Cantor bundles of open arcs (i.e., sets homeomorphic to the product $C \times (0, 1)$ of the Cantor set C and the open interval $(0, 1)$) [Krupski 1982];

   (f) X is homogeneous, contains no proper, nondegenerate, terminal subcontinua and sufficiently small subcontinua of X are not ∞-ods [Krupski 1995, Theorem 3.1];

   (g) X is a homogeneous curve containing an open subset U such that some component of U does not have the disjoint arcs property [Krupski 1995, p. 166];

   (h) X is a homogeneous finitely cyclic (or, equivalently, k-junctioned) curve that is not tree-like and contains no nondegenerate, proper, terminal subcontinua [Krupski et al. XXXXb], [Duda et al. 1991].

   (i) X is openly homogeneous and sufficiently small subcontinua of X are arcs [Prajs 1989];

4. Solenoid $\Sigma(\mathbf{q})$ is a continuous image of $\Sigma(\mathbf{p})$ if and only if the sequence $\mathbf{q} = (q_1, q_2, \dots)$ is a factorant of sequence $\mathbf{p} = (p_1, p_2, \dots)$, i.e., there exists i such that for each $j \geq i$ there is k such that $q_i \cdot \dots \cdot q_j$ is a factor of $p_1 \cdot \dots \cdot p_k$.

   Two solenoids are homeomorphic if and only if each of them is a continuous image of another [Cook 1967], [D. van Dantzig 1930, Satz 8, p. 122].

   There is a family of solenoids of cardinality $2^{\aleph_0}$ such that no member of the family is a continuous image of another.

5. Each monotonemonotone map image of a solenoid X is homeomorphic to X [Krupski 1984b, Theorem 5].

   Each open map transforms X onto a solenoid or onto an arc-like continuum with the property of Kelley and with arcs as proper nondegenerate subcontinua; if the map is a local homeomorphism, then its image is a solenoid [Krupski 1984a].

6. The composant of a solenoid Σ containing e is a one-parameter topological subgroup of Σ, i.e. it is a one-to-one continuous homomorphic image of the additive group of the reals.

7. Any two composants of any two solenoids are homeomorphic [R. de Man 1995].

8. No solenoid can be mapped onto a strongly self-entwined continuum. In particular, it cannot be mapped onto a circle-like plane continuum which is a common part of a descending sequence of circular chains C_i such that C_{i+1} circles n times in C_i clockwisely and then $n-1$ times counter-clockwisely and the first link of C_i contains the closure of the first link of C_{i+1} [Rogers 1971b].

9. No movable continuum (in particular no continuum lying in a surface or a tree-like continuum) can be continuously mapped onto a solenoid. Alternatively, if the first Alexander-Čech cohomology group of a continuum X is finitely divisiblefinitely divisible group, then X cannot be mapped onto a solenoid [Krasinkiewicz 1976, Remark, p. 46, 4.1, 4.9., 5.1], [Krasinkiewicz 1978, Corollary 7.3], [Rogers 1975].

10. Every nonplanar, circle-like continuum has the shape of a solenoid [Krasinkiewicz 1976, remark, p. 46]. Two solenoids have the same shape if and only if they are homeomorphic [Godlewski 1970].

11. Any autohomeomorphism f of $\Sigma(\mathbf{p})$ is isotopic to a homeomorphism g which is induced by a map $(S^1, f_n) \to (S^1, f_n)$ of the inverse sequences which define $\Sigma(\mathbf{p})$ (g can be a group translation, the involution, a power map or its inverse, or compositions of these maps). Maps f and g have equal the topological entropies and are semi-conjugate if the entropy is positive [Kwapisz 2001, Theorems 1–3, pp. 252–253], [D. van Dantzig 1930, Satz 9, p. 125].

    The topological group of all autohomeomorphisms (with the compact-open topology) of a solenoid Σ is homeomorphic (but not isomorphic) to the the product $\Sigma \times l_2 \times Aut(\Sigma)$, where l_2 is the Hilbert space and the group $Aut(\Sigma)$ of all topological group automorphisms of Σ is equipped with the discrete topology and it is equal to $\mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}^n$, or $\mathbb{Z}_2 \oplus_{i=1}^{\infty} \mathbb{Z}$ [Keesling 1972, Theorems 3.1 and 2.4].

12. If the spaces of all autohomeomorphisms of two solenoids are homeomorphic, then the solenoids are isomorphic as topological groups [Keesling 1972, Corollary 3.9].

13. Any map $f : \Sigma(\mathbf{p}) \to \Sigma(\mathbf{q})$ is, for every $\epsilon > 0$, ϵ-homotopic to a map induced by a map $(S^1, f_n) \to (S^1, g_n)$ between inverse sequences defining the corresponding solenoids [Rogers et al. 1971].

14. A $\mathbf{p}$-adic solenoid admits a mean if and only if infinitely many numbers in the sequence $\mathbf{p}$ equal 2 [Krupski XXXXa]. The same condition is equivalent to the non-existence of exactly 2-to-1 map defined on the solenoid [Dębski 1992].

15. The hyperspace of all subcontinua of any solenoid Σ is homeomorphic the cone over Σ [Rogers 1971a], [Nadler 1991, p. 202].

16. The family of all solenoids in the cube I^3 (as a subset of the hyperspace $C(I^3)$) is Borel and not $G_{\delta\sigma}$ [Krupski XXXXc].

4.4 Compactifications of the real line and the real half-line

(a) Waraszkiewicz spirals

4.5 Other examples

Chapter 5

USC DECOMPOSITIONS OF CONTINUA

Chapter 6

CURVES

6.1 Order of a space at a point

(a) Curves of a finite order

(b) Regular continua

(c) Rational continua

(d) Suslinian continua

(e) Urysohn examples $X(n)$, $X(\aleph_0)$; $X(2) = \mathbb{S}^1$

6.2 Case-Chamberlin examples

6.3 Curves without arcs

6.4 Covering defined classes (inverse limit expansions)

(a) Arc-like continua

(b) Circle-like continua (the circle, the Warsaw circle and other elementary examples; the circle of pseudo-arcs)

(c) Tree-like continua (dendrites, dendroids, λ-dendroids; Ingram's example, Bellamy's example)

6.5 Other examples

Chapter 7

FAMILIES OF CONTNINUA

7.1 Universal elements

(a) Universal arc-like continuum

(b) Universal circle-like continuum

(c) Universal hereditarily indecomposable continuum

(d) Universal $\mathcal{P}$-compacta

(e) Universal smooth dendroids

(f) Universal smooth fan

(g) Sierpiński universal plane curve

(h) Menger universal curve

7.2 Common models

7.3 Families of incomparable continua

Chapter 8

PLANE CONTINUA

8.1 Embeddability of continua in surfaces

8.2 Planability in general

8.3 Planability of graphs on of locally connected continua

(a) Kuratowski's graphs

(b) Claytor's examples of nonplanable curves

8.4 Separation of the plane

(a) Common boundary of plane domains

(b) Kuratowski's example

(c) Hereditary cut of the plane (Whyburn's curve)

Chapter 9

MAPPINGS

9.1 Various kinds of mappings and relations between them

Chapter 10

CONTINUA WITH SPECIAL MAPPING PROPERTIES

10.1 Existence of special mappings

(a) Retractions from hyperspaces

(b) Means

(c) Contractibility (also hereditary)

(d) Selectibility

10.2 Self-similar continua

10.3 Self-homeomorphic continua

10.4 Rigid continua

(a) Cook's continua

10.5 Chaotic continua

10.6 Absolute retracts; spaces LC^n and LC^∞

10.7 Absolute retracts for various classes of continua

10.8 Class (W) and other classes defined by mappings

10.9 Homogeneity

10.10 Variations of homogeneity

10.11 Generalized homogeneity

10.12 Embeddability in $\mathbb{R}^n$

10.13 Weakly chainable continua

10.14 Fixed point property

Chapter 11

CONTINUA AS FRACTALS

Chapter 12

DYNAMICAL SYSTEMS

12.1 Attractors

Chapter 13

OTHER PROPERTIES

13.1 Colocal connectedness

13.2 Aposyndesis and its variations

13.3 Property of Kelley (also hereditary)

13.4 Smoothness

13.5 Arc-smoothness

13.6 C^*-smoothness

Chapter 14

NEW FIGURES

14.1 New examples 1

14.1.1 Whyburn's curve W

Whyburn's curve

14.1.2 Pseudo-arc

Pseudo-arc

14.1.3 Zig-zag circle

zig-zag circle

14.1.4 M-continuum

M-continuum

14.1.5 Cantor interaction

Cantor interaction

14.1.6 Pseudo-circle

pseudo-circle

14.1.7 Semicircle continuum

semicircle continuum

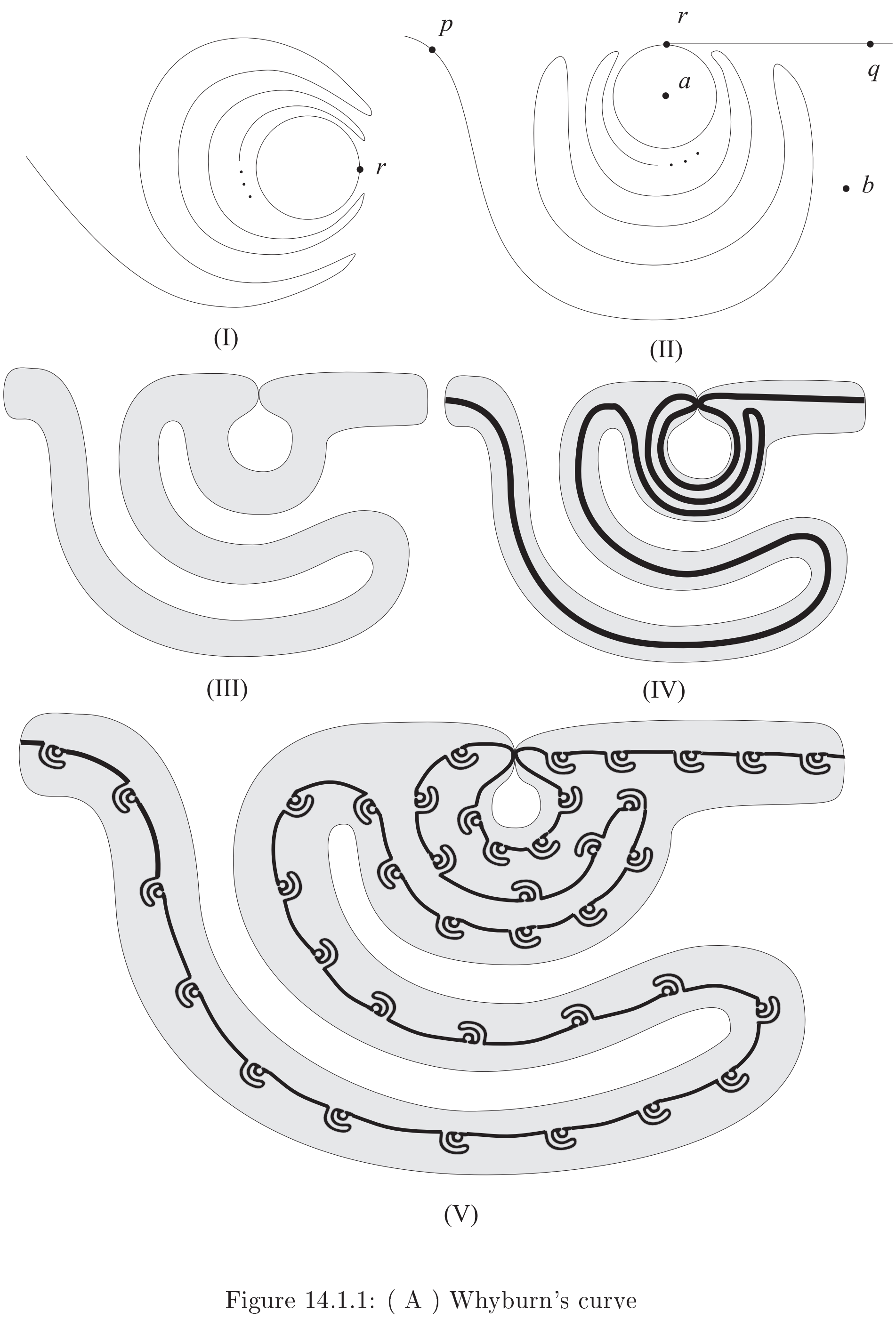


Figure 14.1.1: (A) Whyburn's curve

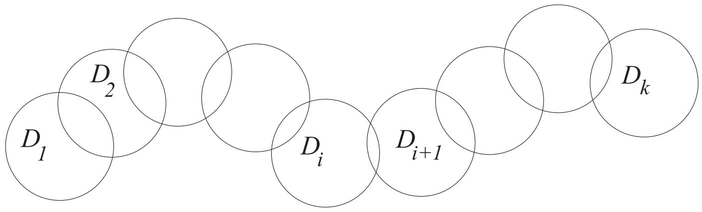


Figure 14.1.2: (A) Pseudo-arc

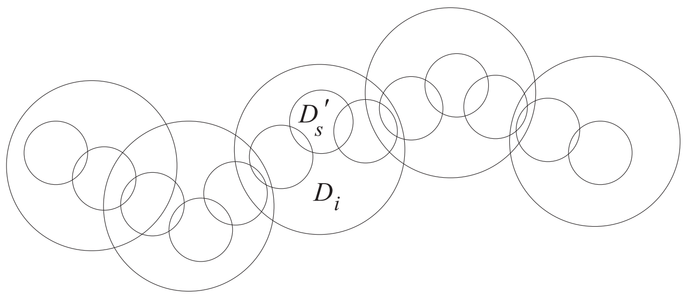


Figure 14.1.2: (B) Pseudo-arc

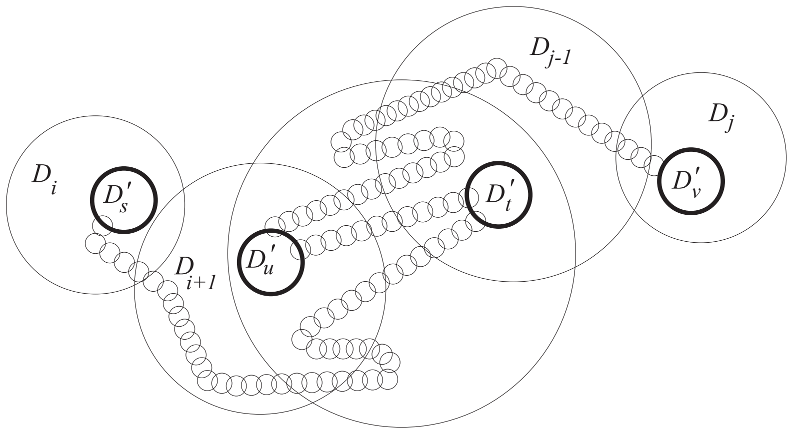


Figure 14.1.2: (C) Pseudo-arc

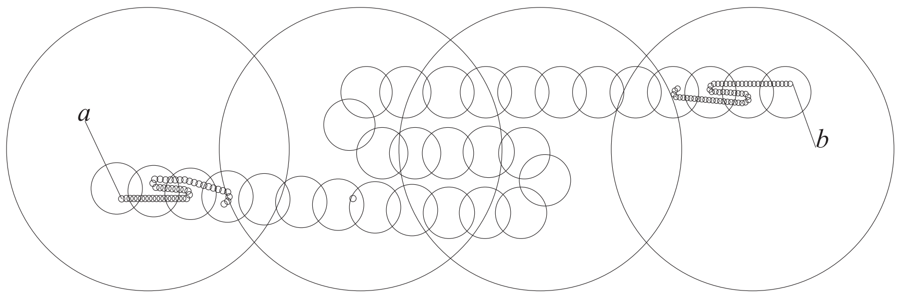


Figure 14.1.2: (D) Pseudo-arc

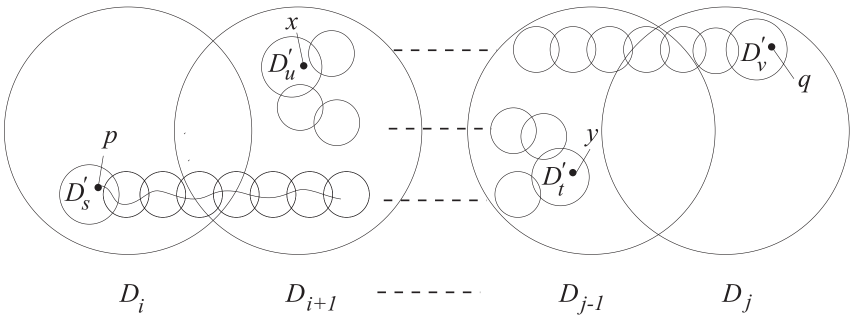


Figure 14.1.2: (E) Pseudo-arc

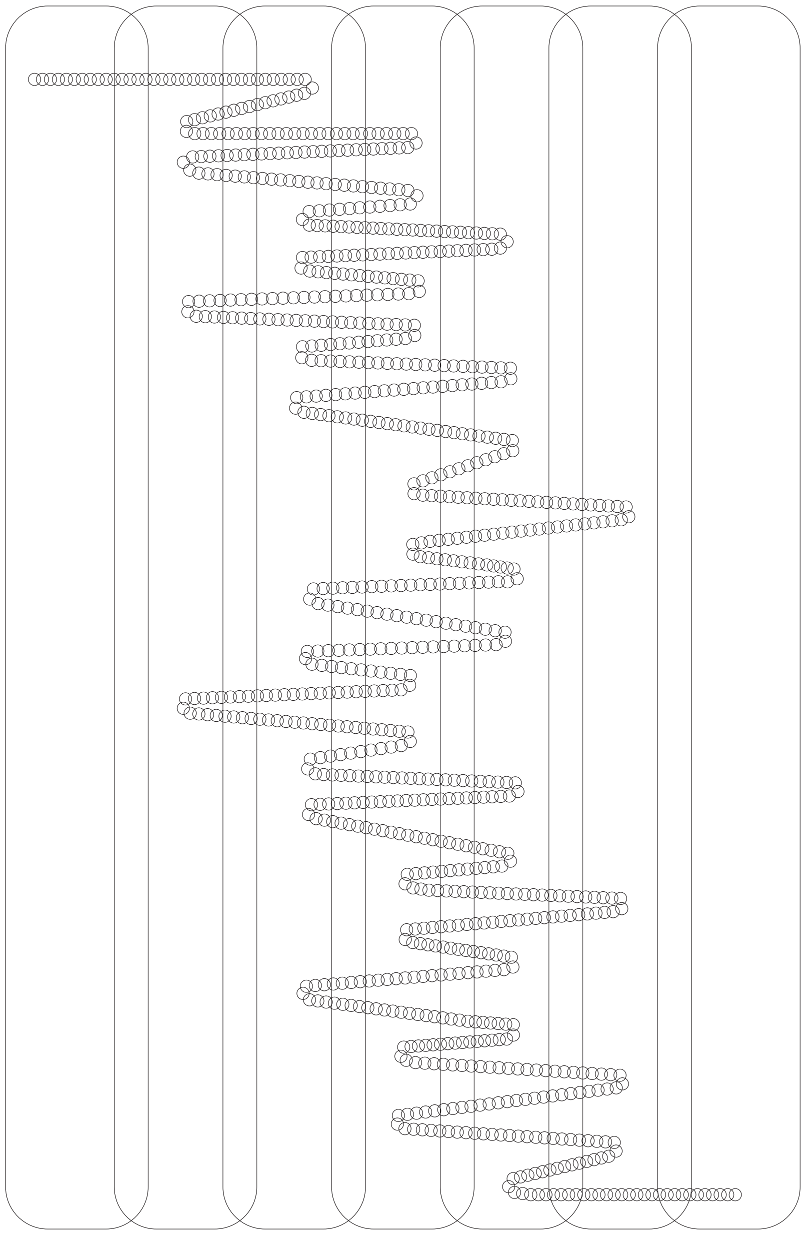


Figure 14.1.2: (F) Pseudo-arc

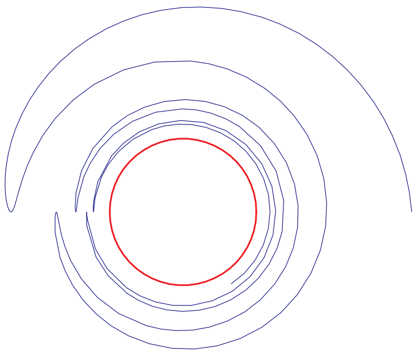


Figure 14.1.3: (A) zig-zag circle

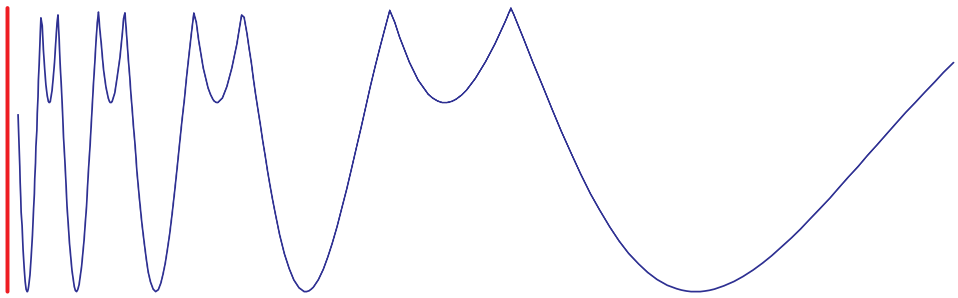


Figure 14.1.4: (A) M-continuum

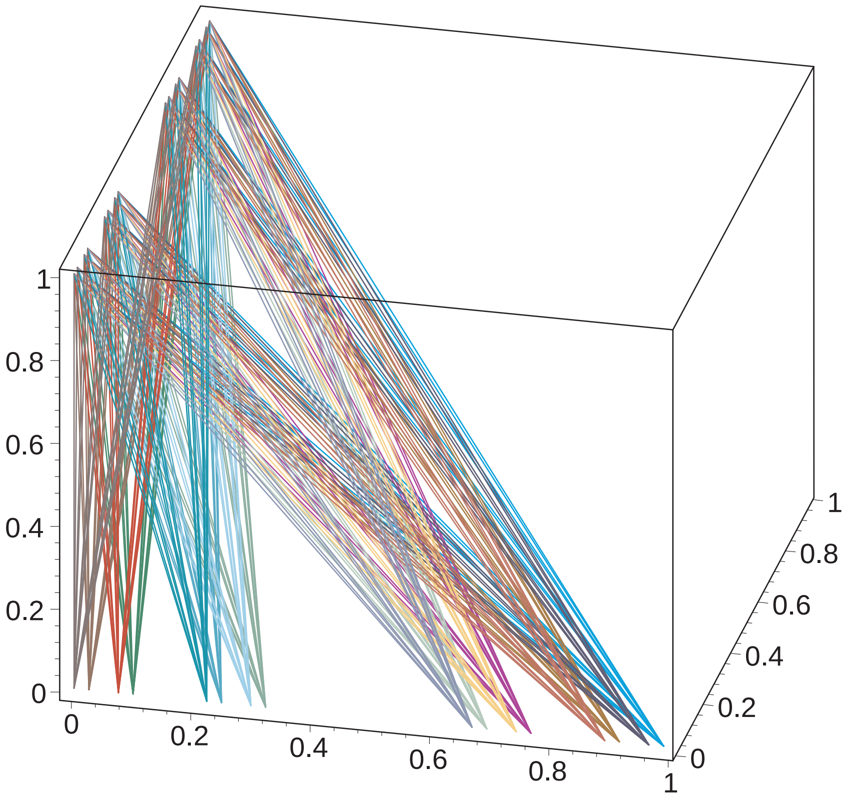


Figure 14.1.5: (A) Cantor interaction

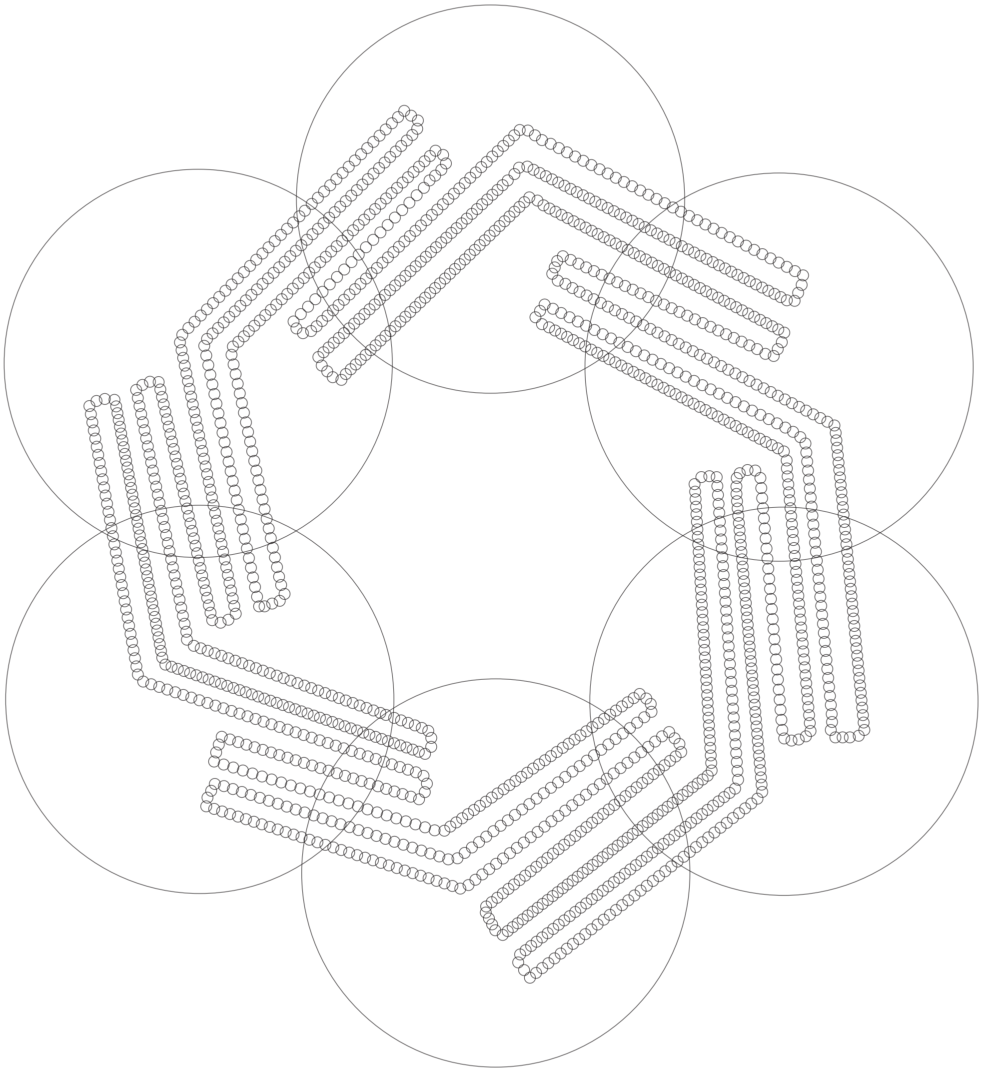


Figure 14.1.6: (A) pseudo-circle

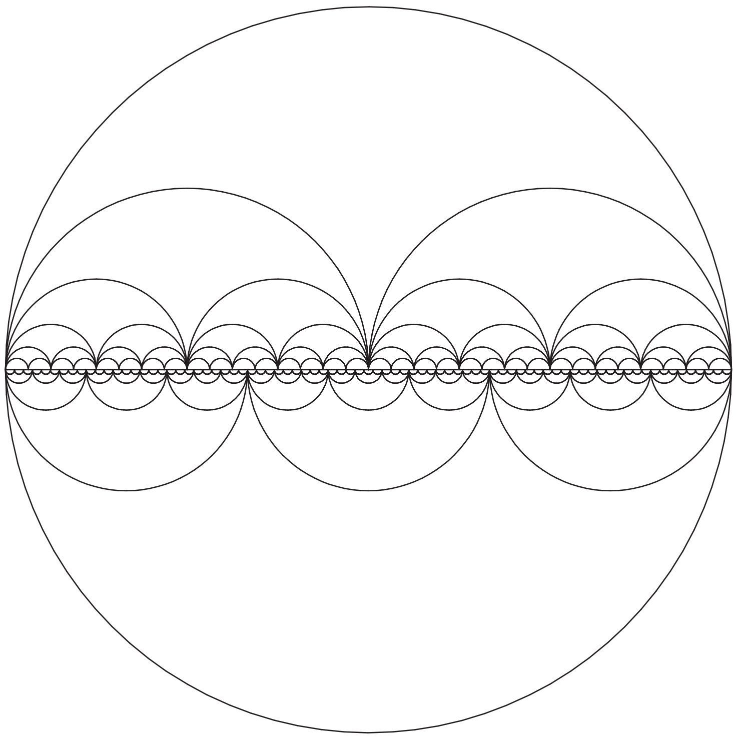


Figure 14.1.7: (A) semicircle continuum

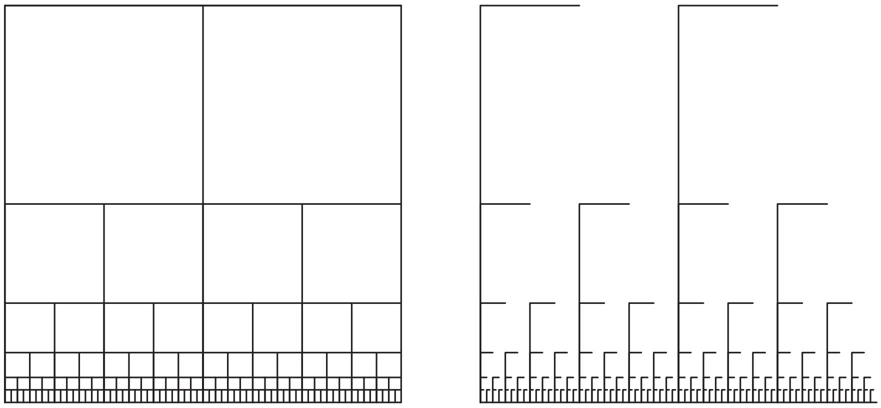


Figure 14.1.8: (A) boxes continuum

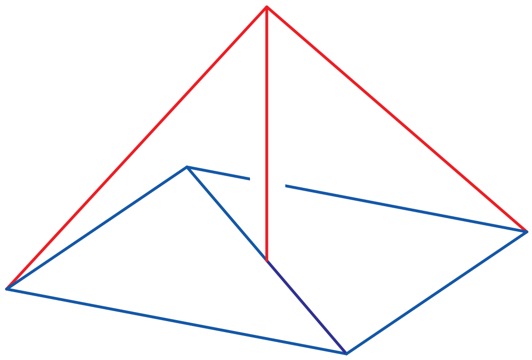


Figure 14.2.1: (A) Graph $K_{3,3}$

14.1.8 Boxes continuum

boxes continuum

14.2 New examples 2

14.2.1 Graph $K_{3,3}$

Graph $K_{3,3}$

14.2.2 Graph K_5

Graph K_5

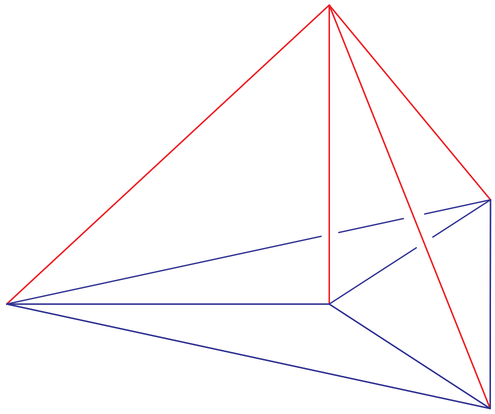


Figure 14.2.2: (A) Graph K_5

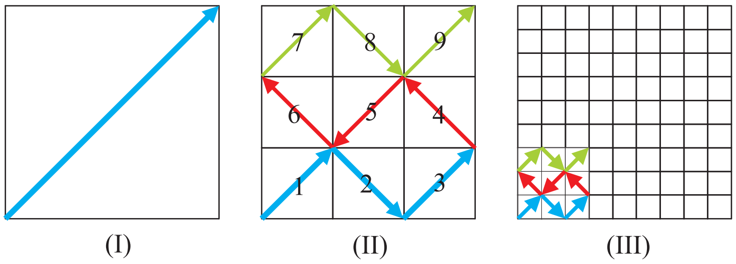


Figure 14.2.3: (A) space filling curve

14.2.3 Space filling curve

space filling curve

14.2.4 Universal arc-like continuum

universal arc-like continuum

14.2.5 Lakes of Vada

lakes of Vada

14.2.6 Borsuk fan

Borsul fan

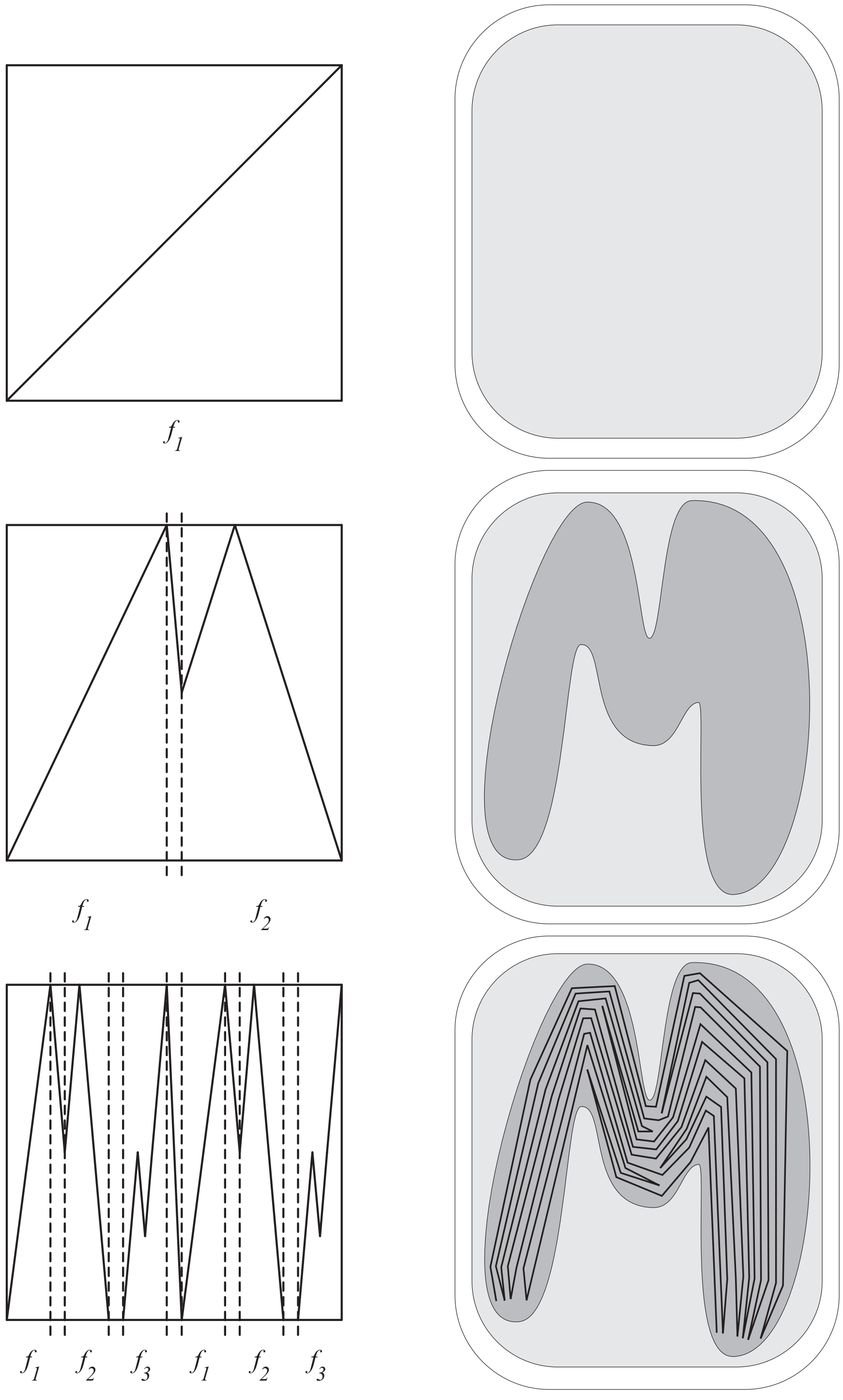
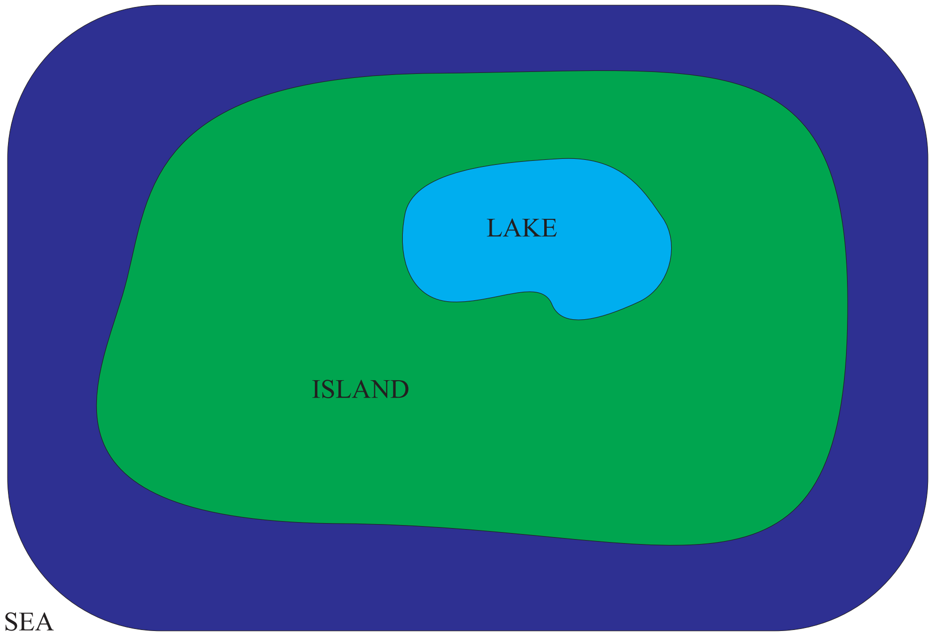
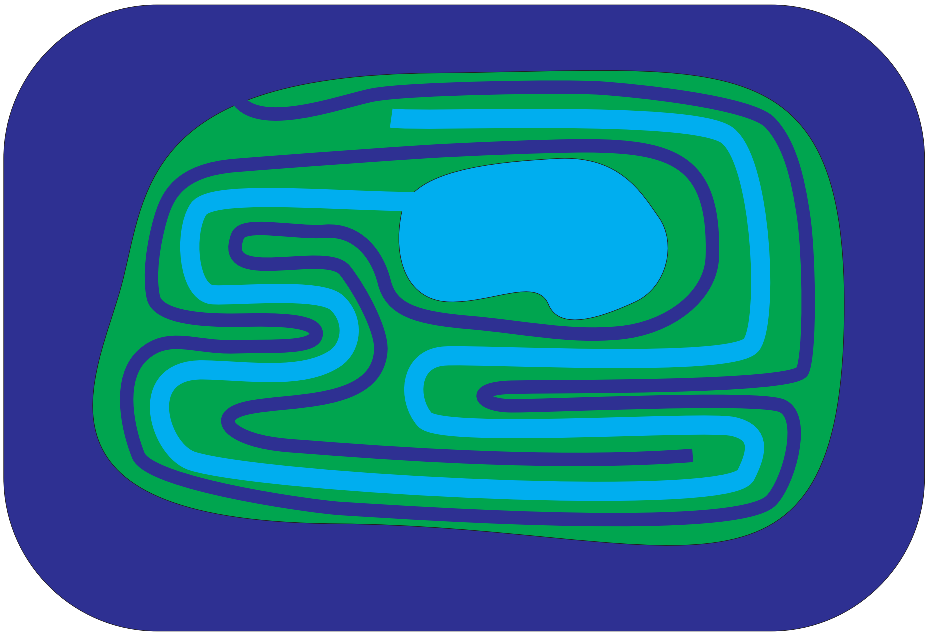


Figure 14.2.4: (A) universal arc-like continuum



(I)



(II)

Figure 14.2.5: (A) lakes of Vada

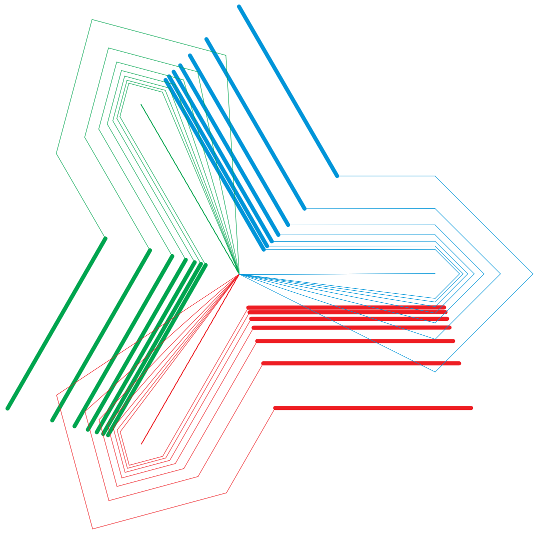


Figure 14.2.6: (A) Borsuk fan

14.2.7 Ingram's atriodic continuum

Ingram's atriodic continuum

14.2.8 Double Warsaw circle

double Warsaw circle

14.2.9 Warsaw circle

Warsaw circle

14.2.10 Crooked Warsaw circle

crooked Warsaw circle

14.3 New examples 3

14.3.1 Young spiral

Young spiral

14.3.2 Young space

Young space

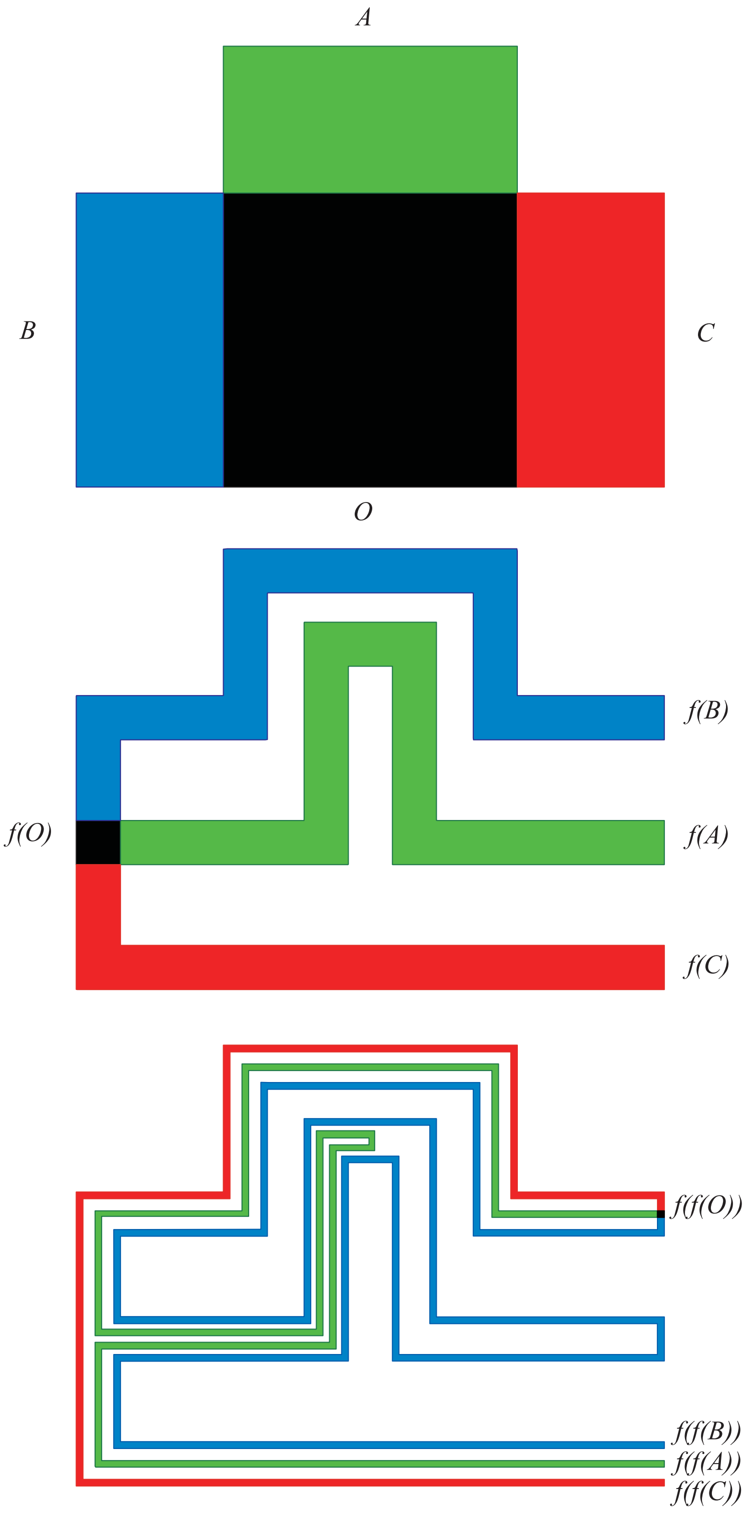


Figure 14.2.7: (A) Ingram's atriodic continuum

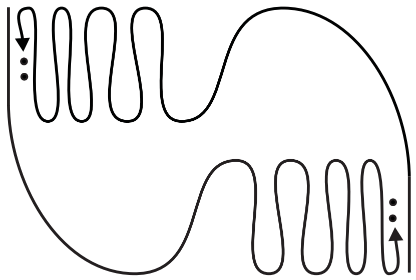


Figure 14.2.8: (A) double Warsaw circle

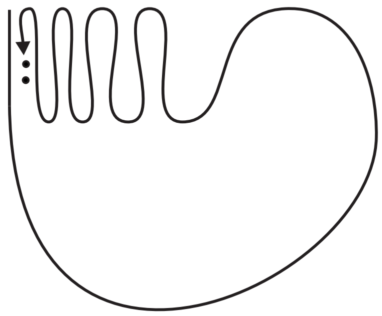


Figure 14.2.9: (A) Warsaw circle

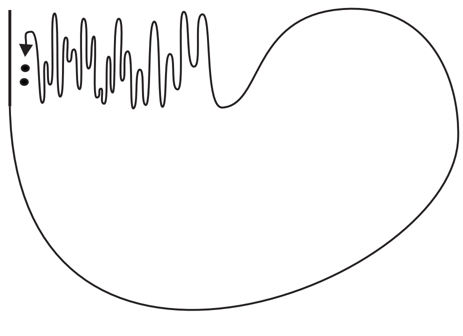


Figure 14.2.10: (A) crooked Warsaw circle

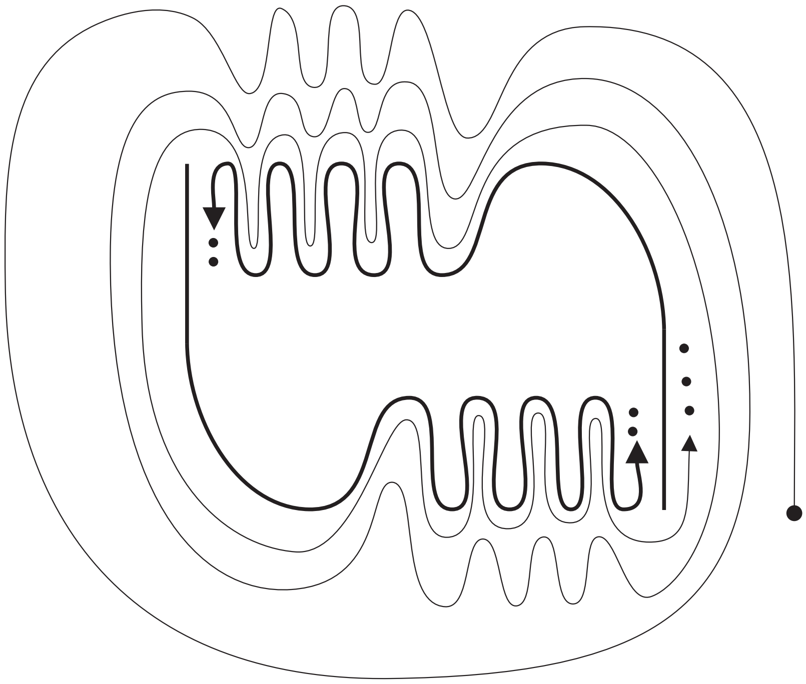


Figure 14.3.1: (A) Young spiral

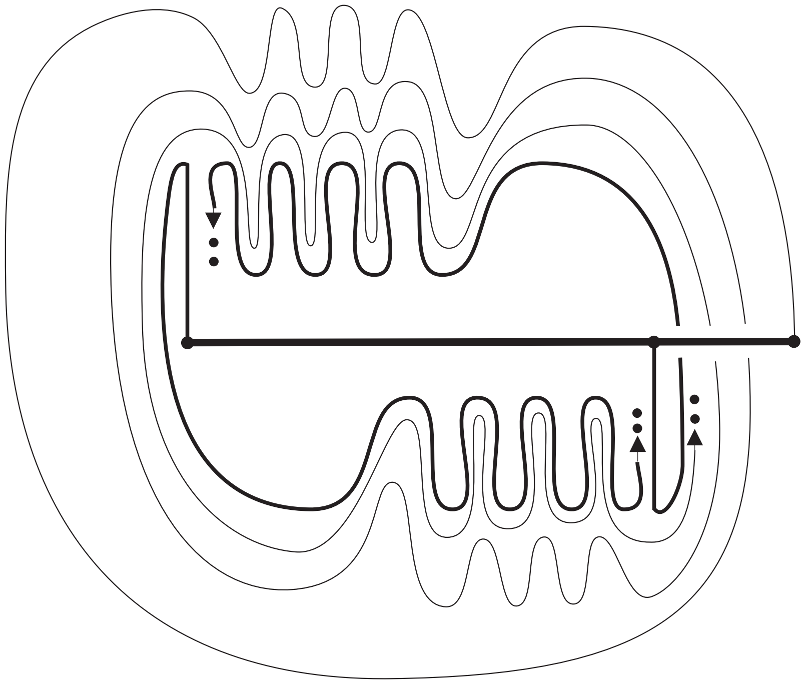


Figure 14.3.2: (A) Young space

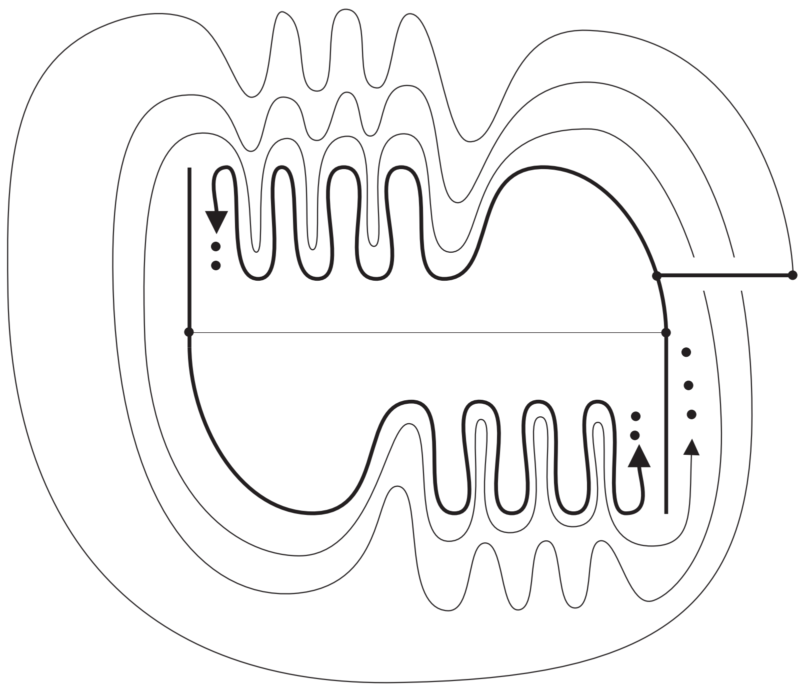


Figure 14.3.3: (A) Bing space

14.3.3 Bing space

Bing space

14.3.4 Universal smooth dendroid

universal smooth dendroid

14.3.5 Claytor's curve A

Claytor's curve A

14.3.6 Claytor's curve B

Claytor's curve B

14.3.7 Sin addiction continuum

sin addiction continuum

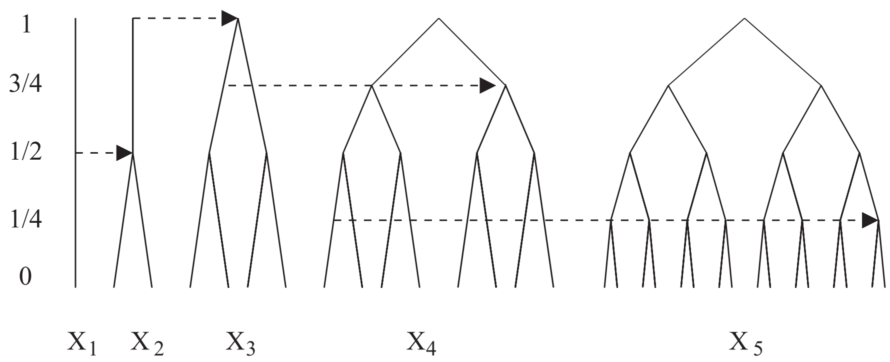


Figure 14.3.4: (A) universal smooth dendroid

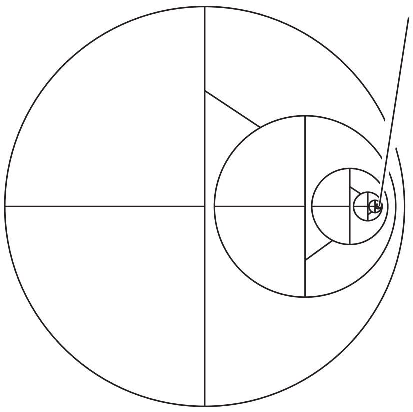


Figure 14.3.5: (A) Claytor's curve A

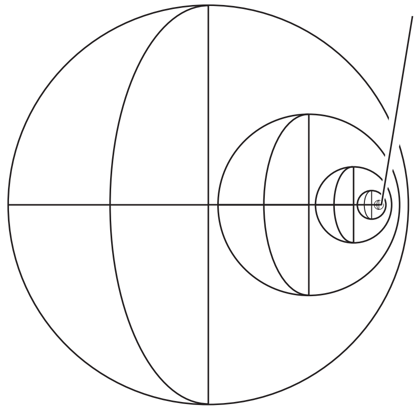


Figure 14.3.6: (A) Claytor's curve B

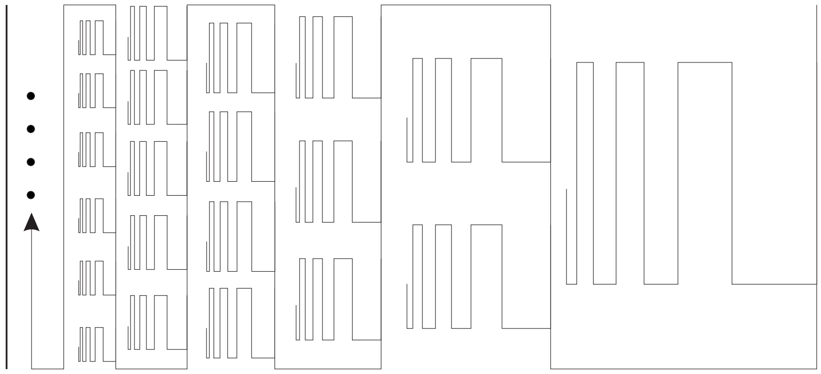


Figure 14.3.7: (A) sin addiction continuum

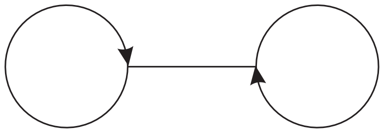


Figure 14.3.8: (A) dumbbell

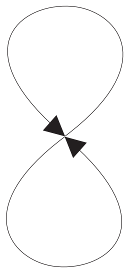


Figure 14.3.9: (A) figure eight

14.3.8 Dumbbell

dumbbell

14.3.9 Figure Eight

figure eight

14.3.10 Noose

noose

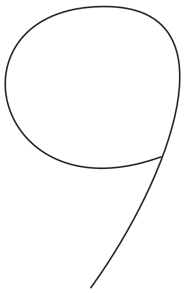


Figure 14.3.10: (A) noose

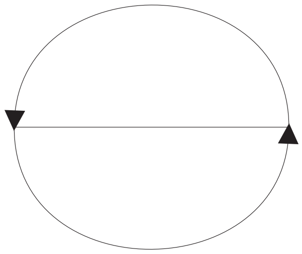


Figure 14.4.1: (A) theta curve

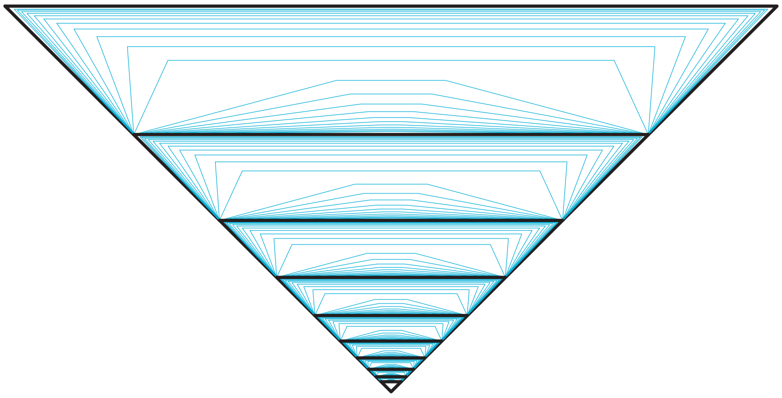


Figure 14.4.2: (A) trapezoid basins continuum

14.4 New examples 4

14.4.1 Theta curve

theta curve

14.4.2 Trapezoid basins continuum

trapezoid basins continuum

14.4.3 Dragon continuum

dragon continuum

L. Fearnley, D. G. Wright : Geometric realization of a Bellamy Continuum, BUll. London Math. Soc. 15(1993), 177-183.

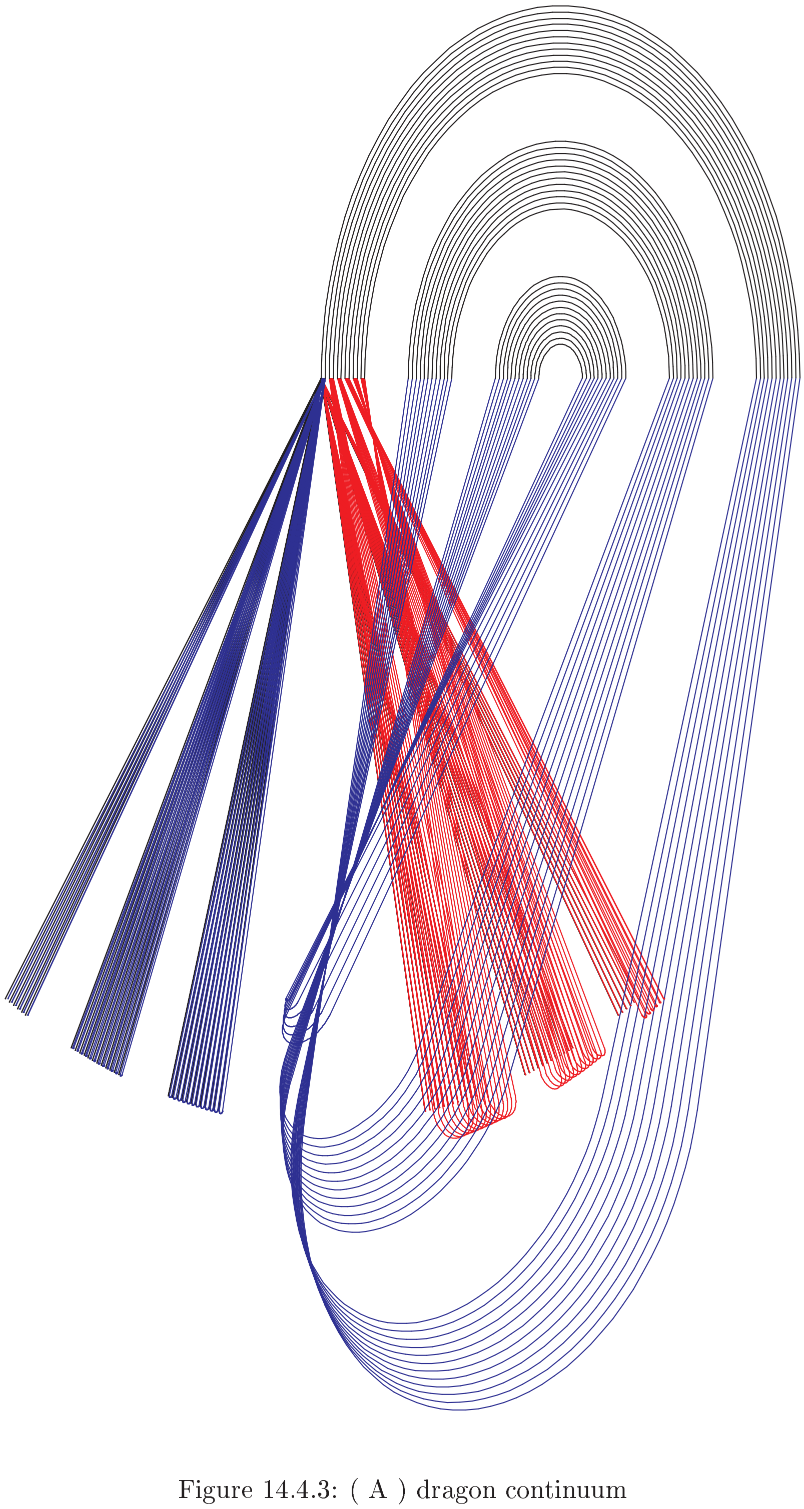
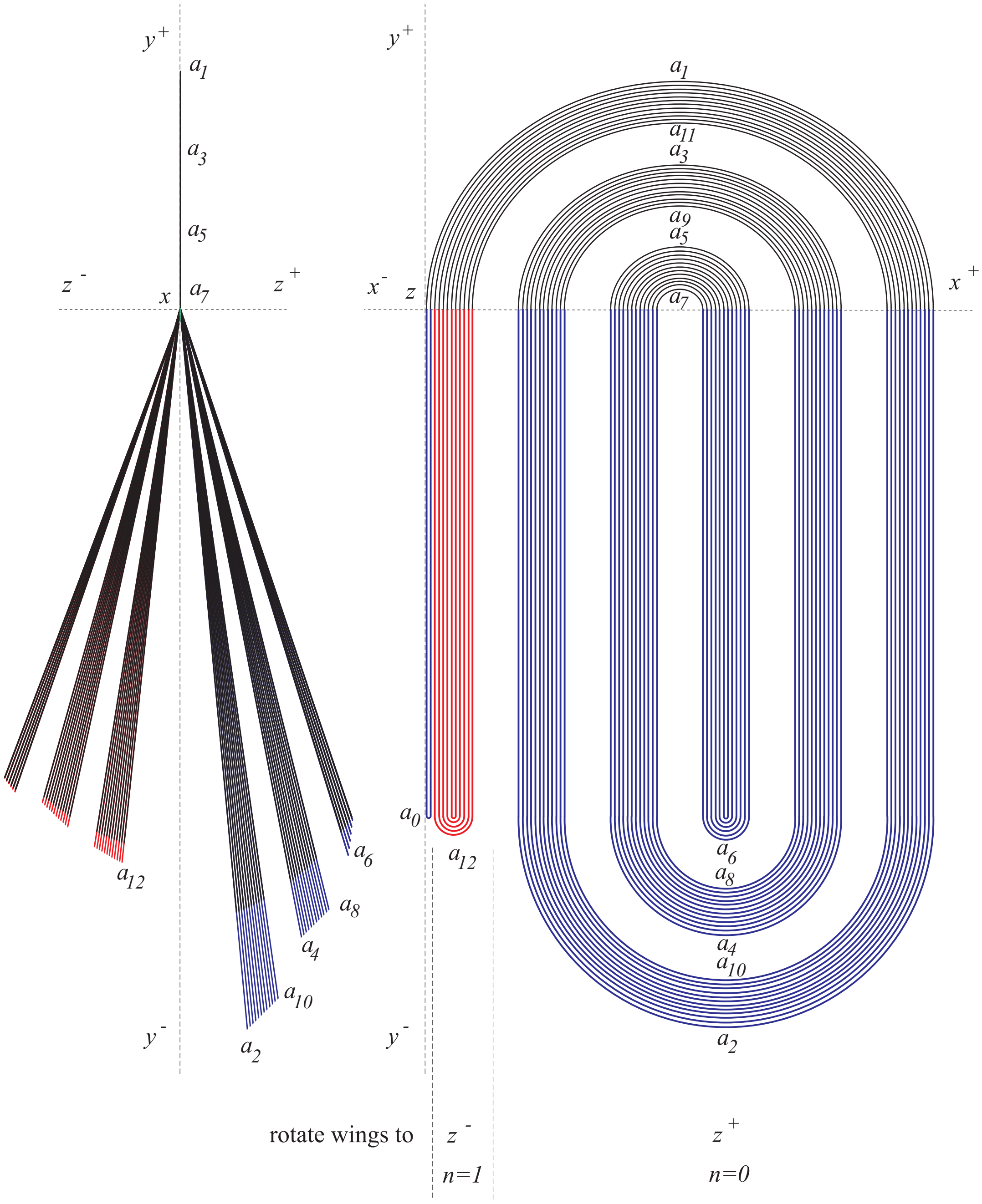


Figure 14.4.3: (A) dragon continuum



Fix point free mapping maps arc $a_n\, a_{n+1}$ to $a_{2n}\, a_{2(n+1)}$

Figure 14.4.3: (B) dragon continuum

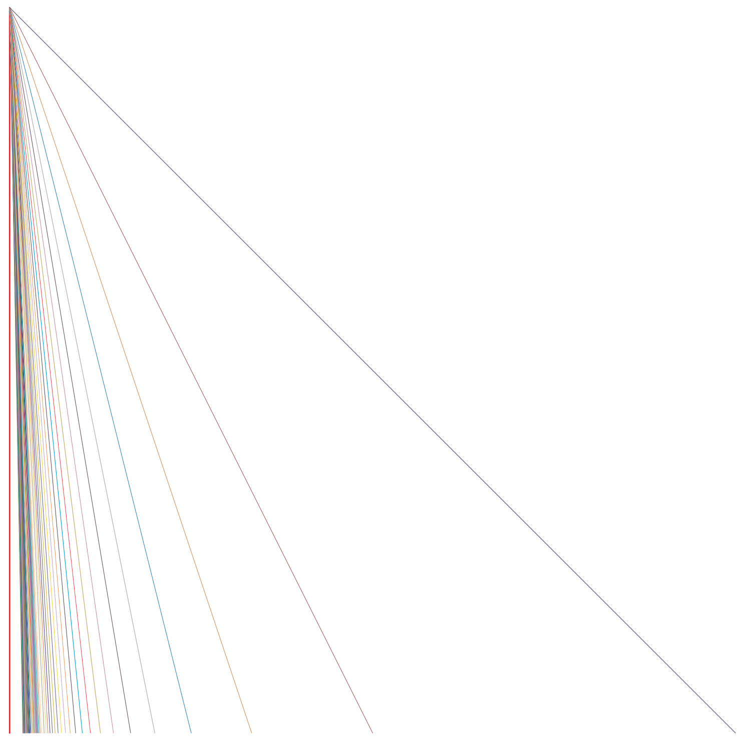


Figure 14.5.1: (A) harmonic fan

14.5 New examples 5

14.5.1 Harmonic fan

harmonic fan

14.5.2 Harmonic shredded fan

harmonic shredded fan

14.5.3 Cantor fan

Cantor fan

14.5.4 Lelek fan

Lelek fan

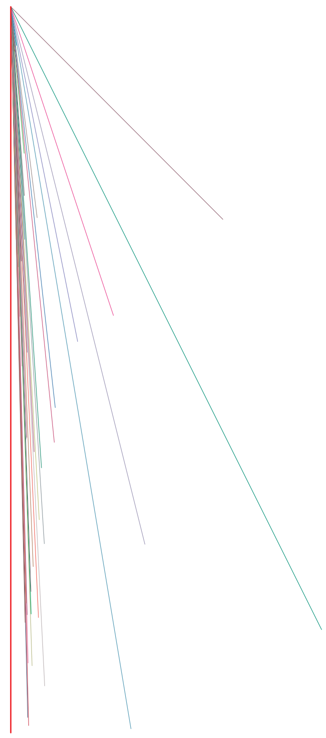


Figure 14.5.2: (A) harmonic shredded fan

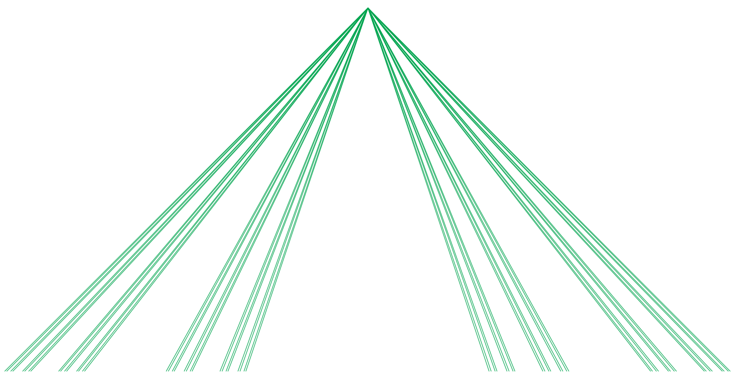


Figure 14.5.3: (A) Cantor fan

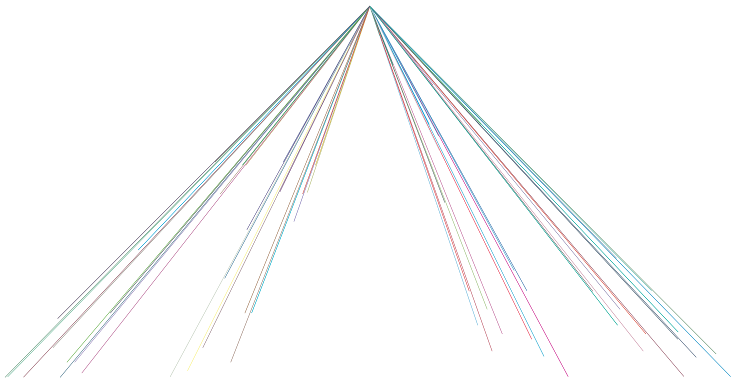


Figure 14.5.4: (A) Lelek fan

14.6 New examples 6

14.6.1 Sierpiński triangle

Sierpiński triangle

14.6.2 Makuchowski umbrella

Makuchowski umbrella

14.6.3 Snake

snake

14.6.4 Cantor snake

Cantor snake

14.6.5 Cruller

cruller

14.7 New examples 7

14.7.1 Cantor comb

Cantor comb

14.7.2 Gehman dendrite on ski

Gehman dendrite on ski

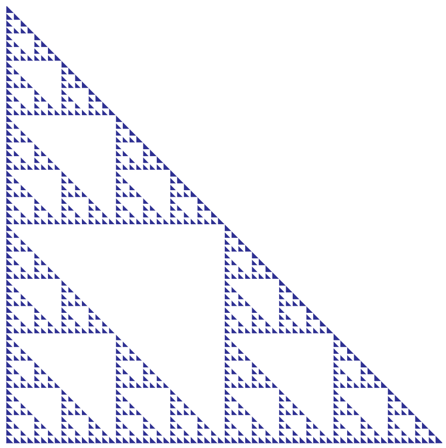


Figure 14.6.1: (A) Sierpiński triangle

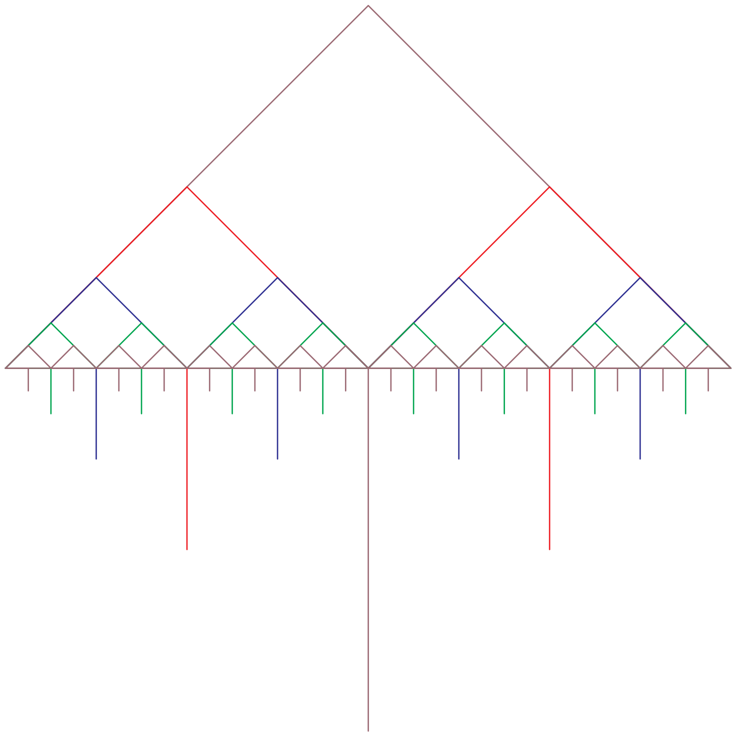


Figure 14.6.2: (A) Makuchowski umbrella

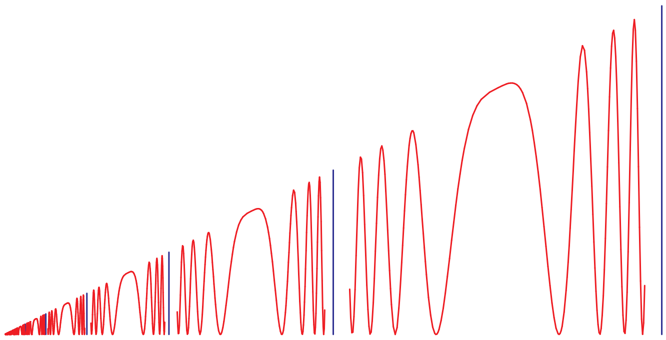


Figure 14.6.3: (A) snake

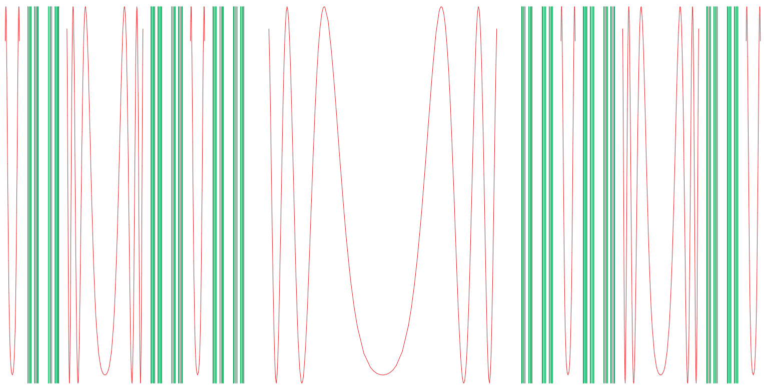


Figure 14.6.4: (A) Cantor snake

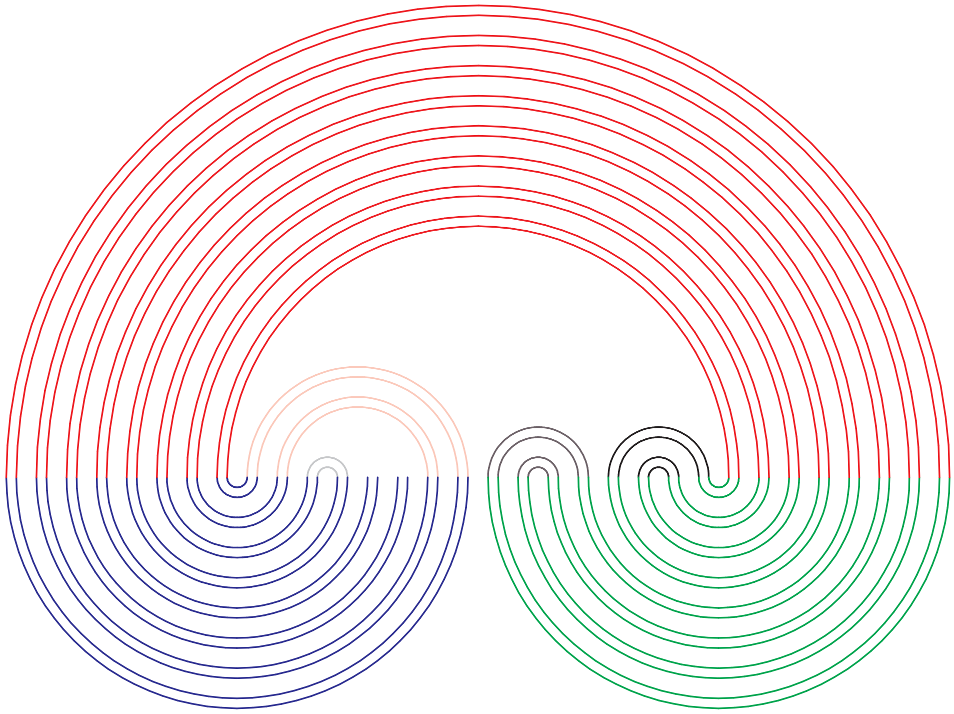


Figure 14.6.5: (A) cruller

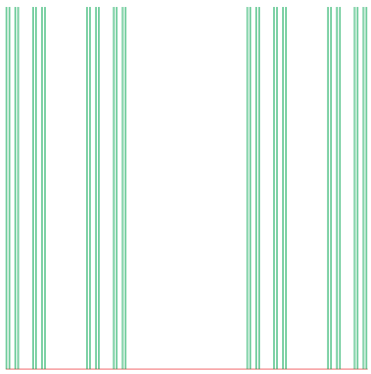


Figure 14.7.1: (A) Cantor comb

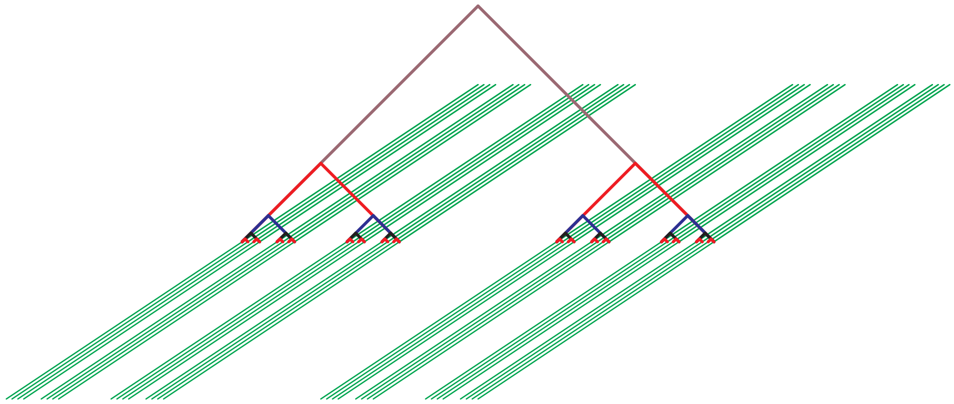


Figure 14.7.2: (A) Gehman dendrite on ski

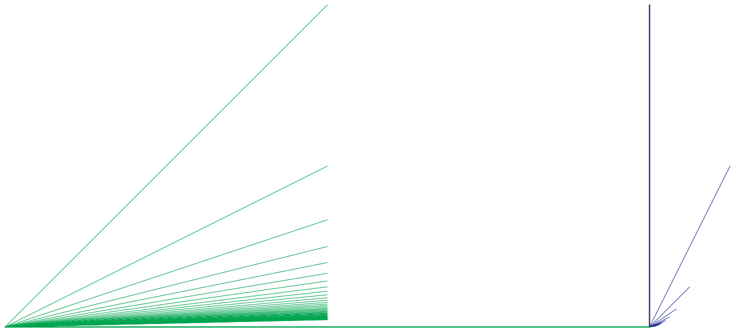


Figure 14.7.3: (A) Makuchowski twin

14.7.3 Makuchowski twin

Makuchowski twin

14.7.4 Cantor function comb

Cantor function comb

14.7.5 Oversteegen's dendroid

Oversteegen's dendroid

Lex G. Oversteegen, *Open retractions and locally confluent mappings of certain continua*, Houston J. Math. **6**,1 (1980), 113–125.

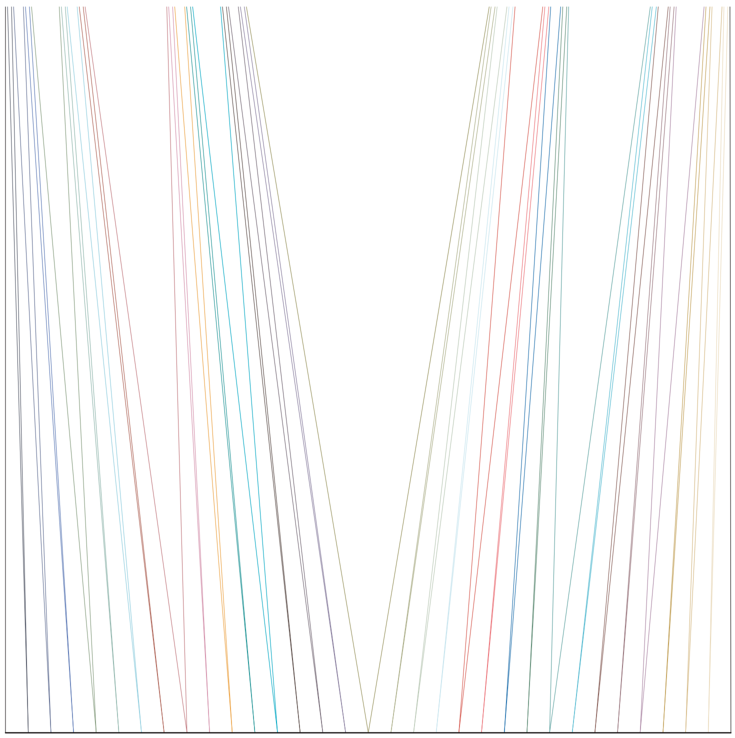


Figure 14.7.4: (A) Cantor function comb

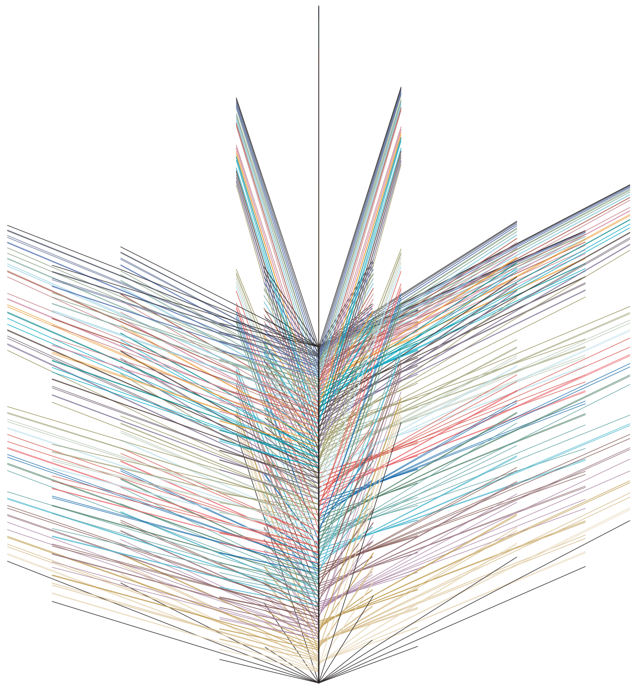


Figure 14.7.5: (A) Oversteegen's dendroid

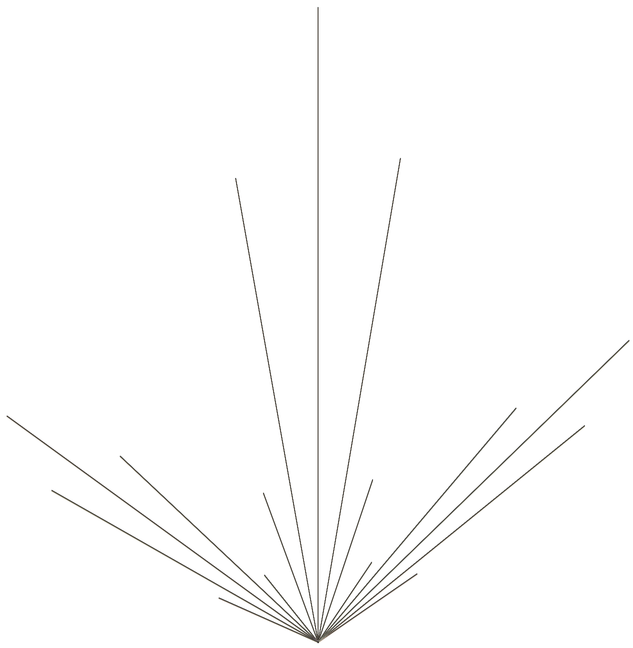


Figure 14.7.5: (B) Oversteegen's dendroid - a bunch of Cantor function combs

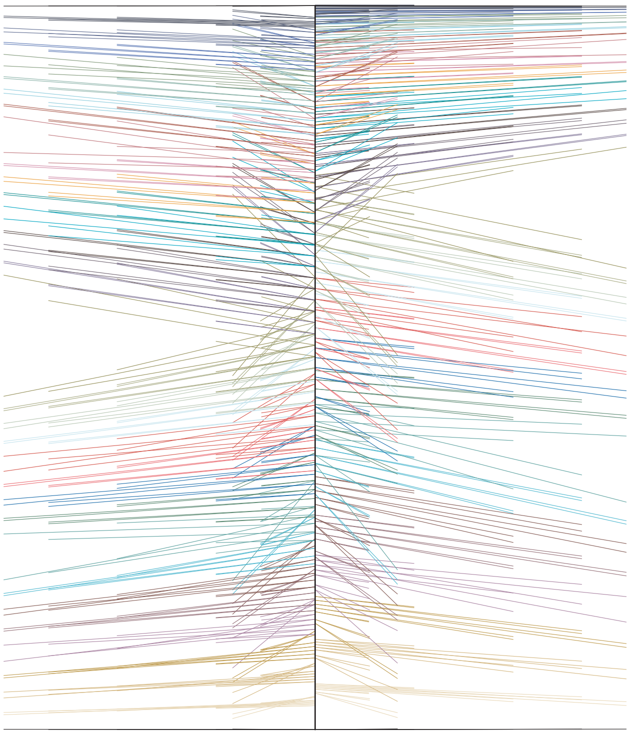


Figure 14.7.5: (C) Oversteegen's dendroid - from above (observe that the Cantor function combs have teeths shifted using the z^2 function, starting from left to right)

Appendix A

Definitions

absolute end point

A point p of a continuum X is called an *absolute end point* of X if $X \setminus \{p\}$ is a composant of X.

absolute retract

An *absolute retract* is a space X such that if X is embedded as a closed subset X' of a space Y, then X' is a retract of Y.

absolutely terminal continuum

A proper subcontinuum K of a continuum X is said to be an *absolutely terminal continuum* of X provided that K is a terminal continuum of each subcontinuum L of X which properly contains K.

acyclic

A continuum X is said to be *acyclic* provided that each mapping from X into the unit circle $\mathbb{S}^1$ is homotopic to a constant mapping, i.e., for all mappings $f : X \to \mathbb{S}^1$ and $c : X \to \{p\} \subset \mathbb{S}^1$ there exists a mapping $h : X \times [0,1] \to \mathbb{S}^1$ such that for each point $x \in X$ we have $h(x,0) = f(x)$ and $h(x,1) = c(x)$.

almost chainable

A continuum M is *almost chainable* if, for each positive number ε, there exists an open cover D of M with mesh ε and a chain C of elements of D and an end link L of C such that no element of $D \setminus C$ intersects any link of C other than L and every point of M is at a distance less than ε from some link of C.

aposyndetic

A connected space X is *aposyndetic at H* with respect to K if there is a closed connected subset of X with H in its interior and not intersecting K, and X is *aposyndetic* if it is aposyndetic at each point with respect to every other point.

A continuum X is said to be *aposyndetic* provided that for each point $p \in X$ and for each $q \in X \setminus \{p\}$ there exists a subcontinuum K of X and an open set U of X such that $p \in U \subset K \subset X \setminus \{q\}$ (see e.g.[Nadler 1992, Exercise 1.22, p. 12]).

arc

An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$.

arc-smooth

Given a continuum X with an arc-structure A, the pair (X, A) (see arc-structure) is said to be *arc-smooth at a point* $v \in X$ provided that the induced function $A_v : X \to C(X)$ defined by $A_v(x) = A(v, x)$ is continuous. Then the point v is called an *initial point* of (X, A). The pair (X, A) is said to be *arc-smooth* provided that there exists a point in X at which (X, A) is arc-smooth. An arbitrary space X is said to be *arc-smooth at a point* $v \in X$ provided that there exists an arc-structure A on X for which (X, A) is arc-smooth at v. The space X is said to be *arc-smooth* if it is arc-smooth at some point (see [Fugate et al. 1981, p. 546]). Note that a dendroid is smooth if and only if it is arc-smooth.

arc-structure

By an *arc-structure* on an arbitrary space X we understand a function $A : X \times X \to C(X)$ such that for every two distinct points x and y in X the set $A(x, y)$ is an arc from x to y and that the following metric-like axioms are satisfied for every points x, y and z in X:

(1) $A(x, x) = \{x\}$;

(2) $A(x, y) = A(y, x)$;

(3) $A(x, z) \subset A(x, y) \cup A(y, z)$,
with equality prevailing whenever $y \in A(x, z)$.

We put (X, A) to denote that the space X is equipped with an arc-structure A (see [Fugate et al. 1981, p. 546]). Note that if there exists an arc-structure on a continuum, then the continuum is arcwise connected.

atomic

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]):
– *atomic* provided that for each subcontinuum K of X such that the set $f(K)$ is nondegenerate we have $K = f^{-1}(f(K))$;

atriodic

a continuum that contains no subcontinuum which is itself a triod is said to be *atriodic*

biconnected

A space is *biconnected* if it is not the sum of two mutually exclusive nondegenerate connected point sets.

bihomogeneous

Suppose X is a space such that, for each two points a and b, there is a homeomorphism of X onto itself such that $h(a) = b$ and $h(b) = a$. Then X is *bihomogeneous*.

Borel sets

The family F of *Borel sets* of a space S is the smallest family satisfying the conditions:

(i) Every closed set belongs to F;
(ii) If X is in F, then $S \setminus X$ is in F;
(iii) The countable intersection of elements of F belongs to F.

branch point

A *branch point* of a continuum is the vertex of a simple triod lying in that continuum.

chainable

A *simple chain* is a finite sequence $L_1, L_2, \cdots, L_n$ of open sets such that L_i intersects L_j if and only if $|i-j| \leq 1$. The terms of the sequence $L_1, L_2, \cdots, L_n$ are called the *links* of the chain. An *ε-chain* is a chain each of whose links has diameter less than ε. a continuum M is *chainable* if, for each positive number ε, M can be covered by an ε-chain.

chaotic

A nondegenerate topological space X is said to be *chaotic* if for any two distinct points p and q of X there exist an open neighborhood U of p and an open neighborhood V of q such that no open subset of U is homeomorphic to any open subset of V.

circle

A *circle* is any space which is homeomorphic to the unit circle.

completely regular

a continuum is *completely regular* if each nondegenerate subcontinuum has a nonempty interior.

composant

If a point p of a continuum X is given, then the *composant of X belonging to p* is defined to be the union of all proper subcontinua of X which contain p. A set is a *composant of X* provided it is the composant of X belonging to p for some point p of X.

confluent

A mapping f of a compact space X onto a compact space Y is *confluent* if, for each continuum C in Y, each component of $f^{-1}(C)$ is mapped onto C by f.

-connected

A topological space (X, τ) is said to be *σ-connected* provided that X is not the union of more than one and at most countably many nonempty, mutually disjoint, closed subsets.

A continuum is *δ-connected* if each two of its points can be connected by an hereditarily decomposable irreducible subcontinuum.

connected im kleinen

A space is *connected im kleinen* if it is aposyndetic at each point with respect to each closed point set not containing that point.

continuous selection of

Let a continuum X, a compact space Y and a function $F : X \to 2^Y$ be given. A function $f : X \to Y$ is called a *continuous selection of* F provided that it is continuous and $f(x) \in F(x)$ for each $x \in X$.

converges homeomorphically to the continuum

The statement that the sequence $M_1, M_2, \cdots$ *converges homeomorphically to the continuum* M means there exists a sequence $h_1, h_2, \cdots$ of homeomorphisms such that, for each positive integer i, h_i is a homeomorphism from M_i onto M and for each positive number ε there exists a positive integer N such that if $j > N$ then, for all x, $\operatorname{dist}(h_j(x), x) < \varepsilon$.

converge 0-regularly

A sequence of sets $\{A_n\}$ contained in a metric space X with a metric d is said to *converge* 0*-regularly* to its limit $A = \operatorname{Lim} A_n$ (see hyperspace) provided that for each $\varepsilon > 0$ there is a $\delta > 0$ and there is an index $n_0 \in \mathbb{N}$ such that if $n > n_0$ then for every two points $p, q \in A_n$ with $d(p,q) < \delta$ there is a connected set $C_n \subset A_n$ satisfying conditions $p, q \in C_n$ and $diam_n < \varepsilon$ (see [Whyburn 1942, Chapter 9, §3, p. 174]).

convex

Given a continuum X with an arc-structure A, a subset Z of X is said to be *convex* provided that for each pair of points x and y of Z the arc $A(x, y)$ is a subset of Z. If Z is a convex subcontinuum of X, then $A|Z \times Z$ is an arc-structure on Z. We define X to be *locally convex at a point* $p \in X$ provided that for each open set U containing p there is a convex set Z such that $p \in \operatorname{int} Z \subset \operatorname{cl} Z \subset U$ (see [Fugate et al. 1981, I.2, p. 548-549]).

continuum

A *continuum* is a nonempty, compact, connected metric space.

contractible

A continuum X is said to be *contractible* provided there is a homotopy $h : X \times [0,1] \to X$ such that for some point $p \in X$ we have $h(x,0) = x$ and $h(x,1) = p$ for each $x \in X$ (see e.g. [Nadler 1978, (16.2), p. 532]).

curve

A *curve* is any one-dimensional continuum.

cut point

Let $S, T)$ be a connected topological space, and let $p \in S$. If $S \setminus \{p\}$ is connected, then p is called a *non-cut point of* S. If $S \setminus \{p\}$ is not connected, then p is called a *cut point of* S.

C^*-smooth

Let X be a continuum. Define $C^* : C(X) \to C(C(X))$ by $C^*(A) = C(A)$. It is known that for any continuum X the function C^* is upper semi-continuous, [Nadler 1978, Theorem 15.2, p. 514], and it is continuous on a dense G_δ subset of $C(X)$, [Nadler 1978, Corollary 15.3, p. 515]. A continuum X is said to be *C^*-smooth at $A \in C(X)$* provided that the function C^* is continuous at A. A continuum X is said to be *C^*-smooth* provided that the function C^* is continuous on $C(X)$, i.e., at each $A \in C(X)$ (see [Nadler 1978, Definition 5.15, p. 517]).

cyclic element

Let a continuum X be locally connected and let $p \in X$. By a *cyclic element* of X containing p we mean either the singleton $\{p\}$ if p is a cut point or an end point of X, or the set consisting of p and of all points $x \in X$ such that no point of X cuts X between p and x, otherwise (see e.g. [Whyburn 1942, Chapter 4, §2, p. 66]).

decomposable

a continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua.

deformation retract of

Let X be a space and let A and B be subspaces of X with $A \subset B$. Then A is called a *deformation retract of* B over X provided that the identity mapping $i_B : B \to B$ is homotopic in X to a retraction $r : B \to A$. Further, A is called a *strong deformation retract of* B over X provided that it is a deformation retract of B over X and the homotopy keeps the points of A fixed throughout the entire deformation of B into A (see e.g. [Dugundji 1966, Definition 6.3, p. 324]).

dendroid

A *dendroid* is an arcwise connected, hereditarily unicoherent continuum.

dendrite

A *dendrite* means a locally connected continuum containing no simple closed curve.

dispersion point

A point p of a nondegenerate connected space X is an *explosion point*, or *dispersion point*, of X provided that $X \setminus \{p\}$ is totally disconnected.

end point

Point of order 1 in a continuum X is called *end point* of X; the set of all end points of X is denoted by $E(X)$.

equivalent

Let $\mathfrak{M}$ be a class of mappings. Two spaces X and Y are said to be *equivalent with respect to* $\mathfrak{M}$ (shortly $\mathfrak{M}$-*equivalent*) if there are two mappings, both in $\mathfrak{M}$, one from X onto Y and the other from Y onto X. If $\mathfrak{M}$ means the class of monotone mappings, we say that X and Y are *monotonely equivalent*.

F-equivalent

If X and Y are continua, we say that Y is F-*equivalent* to X provided there is a mapping in F from X onto Y and a mapping in F from Y onto X.

feebly monotone

Let X and Y be continua. A mapping $f : X \to Y$ is said to be *feebly monotone* provided that if A and B are proper subcontinua of Y such that $Y = A \cup B$, then their inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected.

finitely Suslinean

a continuum X is *finitely Suslinean* if, for each positive number ε, X does not contain infinitely many mutually exclusive subcontinua of diameter greater than ε.

finitely linear

A function $f I \to Y$, where I is a closed interval of the real line, is called *finitely linear* provided there exists a positive integer m such that I can be decomposed, for each $\varepsilon > 0$, into a finite number of closed subintervals $I_1, I_2, \cdots, I_k$ each of length less than ε and with the property that the set $f(I_i)$ meets at most m of the sets $f(I_1), f(I_2), \cdots, f(I_k)$ for $i = 1, 2, \cdots, k$.

fixed point

We say that a function $F : X \to 2^X$ (or a function $F : X \to C(X)$) *has a fixed point* provided that there is a point $x \in X$ such that $x \in F(x)$.

fixed set property

A topological space X is said to have the *fixed set property* for a certain class C of maps of X onto itself provided there exists, for each non-empty closed set A in X, a map f in C such that $f(x) = x$ if and only if x is in A.

fixed point self-homeomorphic

A topological space X is called *fixed point self-homeomorphic* if for any point p, any neighborhood U of p, there is an embedding $h : X \to U$ with $h(p) = p$ and $p \in \operatorname{int} h(X)$.

fixed ball self-homeomorphic

A topological space X is called *fixed ball self-homeomorphic* if for any point p, any neighborhood U of p, there is a neighborhood V of p with $V \subseteq U$ and an embedding $h : X \to U$ satisfying $h|_V = \mathrm{id}_V$.

free arc

A *free arc* in X is any subset of X of the form $A \setminus \{x, y\}$ where A is an arc in X with end points x and y and $A \setminus \{x, y\}$ is open in X.

graph

A *graph* is a continuum which can be written as the union of finely many arcs any two of which are either disjoint or intersect only in one or both their end points.

half-ray curve

a continuum is a *half-ray curve* if it is a continuous 1-1 image of the nonnegative reals.

hereditary

A property of a continuum X is said to be *hereditary* if every subcontinuum of X has the property. In particular, a continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected.

hereditarily

Let $\mathfrak{M}$ be a class of mappings between continua. A mapping $f : X \to Y$ between continua is said to be *hereditarily* $\mathfrak{M}$ provided that its restriction to any subcontinuum of the domain X is in $\mathfrak{M}$ (see [Maćkowiak 1979, Chapter 4, Section B, p. 16]).

hereditarily decomposable

a continuum X is said to be *hereditarily decomposable* provided that each non-degenerate subcontinuum of X (as well as X itself) is decomposable.

hereditarily equivalent

a continuum is *hereditarily equivalent* if it is homeomorphic to each of its non-degenerate subcontinua.

hereditarily indecomposable

a continuum X is said to be *hereditarily indecomposable* provided that each of its subcontinua is indecomposable, that is, for each subcontinuum $C \subset X$ and for every continua A and B such that $A \cup B = C$ we have either $A = C$ or $B = C$.

hereditarily locally connected

A continuum X is said to be *hereditarily locally connected*, written *hlc*, provided that every subcontinuum of X is a Peano Continuum.

hereditarily unicoherent

A connected topological space is said to be *hereditarily unicoherent* provided that each of its closed, connected subsets is unicoherent.

homogeneous

Let $\mathfrak{M}$ be a class of mappings. A space X is said to be *homogeneous with respect to* $\mathfrak{M}$ (or shortly $\mathfrak{M}$-*homogeneous*) provided that for every two points p and q of X there is a surjective mapping $f : X \to X$ such that $f(p) = q$ and $f \in \mathfrak{M}$. If $\mathfrak{M}$ is the class of homeomorphisms, we get the concept of a *homogeneous* space.

HU-terminal

A subcontinuum K of a hereditarily unicoherent continuum X is said to be a *HU-terminal continuum* of X provided that K is contained in an irreducible subcontinuum of X and for every irreducible subcontinuum I of X containing K there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$.

hyperspace

Given a continuum X with a metric d, we let 2^X to denote the *hyperspace* of all nonempty closed subsets of X equipped with the *Hausdorff metric* H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\} \quad \text{for } A, B \in 2^X$$

(see e.g. [Nadler 1978, (0.1), p. 1 and (0.12), p. 10]). If $H(A, A_n)$ tends to zero as n tends to infinity, we put $A = \operatorname{Lim} A_n$. Further, we denote by $F_1(X)$ the hyperspace of singletons of X, and by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. Since X is homeomorphic to $F_1(X)$, there is a natural embedding of X into $C(X)$, and so we can write $X \subset C(X) \subset 2^X$. Thus one can consider a retraction from either $C(X)$ or 2^X onto X.

indecomposable

a continuum X is said to be *indecomposable* provided that X cannot be written as the union of two proper subcontinua.

induced mapping

Let $f : X \to Y$ be a mapping between continua. Then the *induced mapping* $C(f) : C(X) \to C(Y)$ is defined by $C(f)(A) = f(A)$, where A in the left member of the equality means an element of $C(X)$, while in the right one it is understood as a subcontinuum of X (see [Nadler 1978, (0.49), p. 23]).

inverse limit

The reader can find necessary information on the inverse limits of inverse sequences e.g. in Section 2 of the second chapter of [Nadler 1992], as well as in [Kuratowski 1968].

irreducible

Let X be a continuum and $A \subset X$. Then, X is said to be *irreducible about* A provided no proper subcontinuum of X contains A. a continuum X is said to be *irreducible* provided that X is irreducible about $\{p, q\}$ for some $p, q \in X$.

light

A mapping is *light* if each point inverse is totally disconnected.

like

If X is a metric space, a mapping f from X to a space Y is an ε*-map* if, for each point y of Y, $diam f^{-1}(y)) \leq \varepsilon$. If C is a collection of continua, a continuum M is C*-like* if, for every positive number ε, there exists an ε-map of M onto an element of C. In particular, a continuum is *tree-like* if, for some collection C of trees, M is C-like.

A concept of a tree-like continuum can be defined in several (equivalent) ways. One of them is the following. A continuum X is said to be *tree-like* provided that for each $\varepsilon > 0$ there is a tree T and a surjective mapping $f : X \to$

T such that f is an ε-mapping (i.e., $diam^{-1}(y) < \varepsilon$ for each $y \in T$). Let us mention that a continuum X is tree-like if and only if it is the inverse limit of an inverse sequence of trees with surjective bonding mappings. Compare e.g. [Nadler 1992, p. 24].

Using a concept of a nerve of a covering, one can reformulate the above definition saying that a continuum X is be tree-like provided that for each $\varepsilon > 0$ there is an ε-covering of X whose nerve is a tree.

Finally, the original definition using tree-chains can be found e.g. in Bing's paper [Bing 1951, p. 653].

locally

Let $\mathfrak{M}$ be a class of mappings between compact spaces. A surjective mapping $f : X \to Y$ between continua is said to be *locally* $\mathfrak{M}$ provided that for each point $x \in X$ there is a closed neighborhood V of x such that $f(V)$ is is a closed neighborhood of $f(x)$ and that the restriction $f|V$ is in $\mathfrak{M}$ (see [Maćkowiak 1979, Chapter 4, Section C, p. 18]).

locally confluent

A mapping f from a space X onto a space Y is *locally confluent* if, for each point y of Y, there is an open set U containing y such that $f^{-1}(\overline{U})$ is confluent.

local homeomorphism

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]):
– a *local homeomorphism* provided that for each point $x \in X$ there exists a an open neighborhood U of x such that $f(U)$ is an open neighborhood of $f(x)$ and that f restricted to U is a homeomorphism between U and $f(U)$.

local separating point

A point p of a locally compact separable metric space L is a *local separating point* of L provided there exists an open set U of L containing p and two points x and y of the component containing p of U such that $U \setminus \{p\}$ is the sum of two mutually separated point sets, one containing x and the other containing y.

MO-mapping

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]):

– an *MO-mapping* provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$ such that f_1 is open and f_2 is monotone.

monotone map

A continuous function $f : X \to Y$ is said to be *monotone* provided $f^{-1}(y)$ is connected for each $y \in Y$.

mutually aposyndetic

a continuum M is *mutually aposyndetic* if, for each two points A and B of M, there exist mutually exclusive subcontinua H and K of M containing A and B, respectively, in their interiors. a continuum is *strictly non-mutually aposyndetic* if each two of its subcontinua with interiors intersect.

natural projection

Let continua X and Y be given. A mapping $f : X \times Y \to X$ is called the *natural projection* provided that it is defined by $f((x, y)) = x$.

neat

A dendroid X is said to be *neat* provided that each one of its subdendroids has no improper shore point (see [Neumann-Lara et al. 1993, p. 939]).

Let $\mathfrak{M}$ be a class of mappings. A class $\mathfrak{M}$ of mappings is said to be *neat* provided that if all homeomorphisms are in $\mathfrak{M}$ and the composition of any two mappings in $\mathfrak{M}$ is also in $\mathfrak{M}$.

near homeomorphism

A mapping $f : X \to Y$ is called a *near homeomorphism* provided that for each $\varepsilon > 0$ there is a homeomorphism $h : X \to Y$ such that $\sup\{d(f(x), h(x)) : x \in X\} < \varepsilon$.

OM-mapping

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]):
– an *OM-mapping* provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$ such that f_1 is monotone and f_2 is open.

open map

A mapping f of a space X into a space Y is *open* if, for each open A in X the image $f(A)$ is open in Y.

openly minimal

We say that a dendrite X is *openly minimal* provided that every open image of X can be openly mapped onto X.

orbit

Given a space X let $\mathcal{H}(X)$ stand for the group of autohomeomorphisms of X. If a point $p \in X$ is fixed, then $\{h(p) \in X : h \in \mathcal{H}(X)\}$ is called an *orbit of* p. Orbits of points of X either are mutually disjoint or coincide, and their union is the whole X.

order

Let (X, T) be a topological space, and let $A \subset X$. Let β be a cardinal number. We say that A *is of order less than equal to* β in X, written $\text{ord}(A, X) \leq \beta$, provided that for each $U \in T$ such that $A \subset U$, there exists $V \in T$ such that $A \subset V \subset U$ and $|Bd(V)| \leq \beta$.

We say that A *is of order* β *in* X, written $\text{ord}(A, X) = \beta$, provided that $\text{ord}(A, X) \leq \beta$ and $\text{ord}(A, X) \not\leq \alpha$ for any cardinal number $\alpha < \beta$.

A concept of an *order* of a point p in a continuum X (in the sense of Menger-Urysohn), written $\text{ord}\,(p, X)$, is defined as follows. Let $\mathfrak{n}$ stand for a cardinal number. We write:

$\text{ord}\,(p, X) \leq \mathfrak{n}$ provided that for every $\varepsilon > 0$ there is an open neighborhood U of p such that $diam \leq \varepsilon$ and $\text{card}\,\text{bd}\,U \leq \mathfrak{n}$;

$\text{ord}\,(p, X) = \mathfrak{n}$ provided that $\text{ord}\,(p, X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\text{ord}\,(p, X) \leq \mathfrak{m}$ does not hold;

$\text{ord}\,(p, X) = \omega$ provided that the point p has arbitrarily small open neighborhoods U with finite boundaries $\text{bd}\,U$ and $\text{card}\,\text{bd}\,U$ is not bounded by any $n \in \mathbb{N}$.

Thus, for any continuum X we have

$$\text{ord}\,(p, X) \in \{1, 2, \ldots, n, \ldots, \omega, \aleph_0, 2^{\aleph_0}\}$$

(convention: $\omega < \aleph_0$); see [Kuratowski 1968, §51, I, p. 274].

Let a dendroid X and a point $p \in X$ be given. Then p is said to be a *point of order at least* $\mathfrak{m}$ *in the classical sense* provided that p is the center of an $\mathfrak{m}$-od contained in X. We say that p is a *point of order* $\mathfrak{m}$ *in the classical sense* provided that $\mathfrak{m}$ is the minimum cardinality for which the above condition is satisfied (see [Charatonik 1962, p. 229]).

order preserving mapping

For each point p of a continuum X equipped with an arc-structure $A : X \times X \to C(X)$ we define a partial order $\leq_p$ by letting $x \leq_p y$ whenever $A(p, x) \subset A(p, y)$. Let X and Y be continua with fixed arc-structures A and B, respectively. We say that a surjective mapping $f : X \to Y$ is a *$\leq_p$-mapping* provided that $x \leq_p y$ in X implies that $f(x) \leq_{f(p)} f(y)$ in Y. If, in addition, $Y \subset X$, $B = A|(Y \times Y)$, and f is a retraction, then f is called a *$\leq_p$-retraction* (or *$\leq_p$-preserving retraction*). The concept of a *$<_p$-mapping* is defined in a similar manner (with $f(x) \neq f(y)$ implied by $x \neq y$). For *order preserving mapping*s see e.g. [Fugate et al. 1981, I.7, p. 553].

ordinary point

Point of order 2 is called *ordinary point* of X; the set of all ordinary points of X is denoted by $O(X)$.

periodic point of

Let a continuum X and a mapping $f : X \to X$ be given. For each natural number n denote by f^n the n-th iteration of f. A point $p \in X$ is called a *periodic point of* f provided that there is $n \in \mathbb{N}$ such that $f^n(p) = p$ The set of periodic points of a mapping $f : X \to X$ are denoted by $P(f)$.

periodic-recurrent property

The set of periodic points and of recurrent points of a mapping $f : X \to X$ are denoted by $P(f)$ and $R(f)$ respectively. Clearly $P(f) \subset R(f)$. A continuum X is said to have the *periodic-recurrent property* (shortly *PR-property*) provided that for every mapping $f : X \to X$ the equality $\operatorname{cl} P(f) = \operatorname{cl} R(f)$ holds.

pointwise self-homeomorphic

A topological space X is called *pointwise self-homeomorphic at a point $x \in X$* if for any neighborhood U of x there is a set V such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *pointwise self-homeomorphic* if it is pointwise self-homeomorphic at each of its points.

pseudo-arc

A *pseudo-arc* is the only (up to homeomorphism) hereditarily indecomposable arc-like continuum.

pseudo-confluent

A mapping from X to Y is *pseudo-confluent* if every irreducible continuum in Y is the image of a continuum in X.

property of Kelley

A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see e.g. [Nadler 1978, Definition 16.10, p. 538]).

quasi-monotone

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]) *quasi-monotone* provided that for each subcontinuum Q of Y with the nonempty interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q.

radially convex at

Given a continuum X with an arc-structure A and a point $p \in X$, a metric d on X is said to be *radially convex at* p provided that $d(p, z) = d(p, y) + d(y, z)$ for every points $y, z \in X$ with $y \in A(p, z)$ (see [Fugate et al. 1981, I.4, p. 551]).

ramification point

Point of order at least 3 is called *ramification point* of X; the set of all ramification points of X is denoted by $R(X)$.

rational

A continuum is *rational* if every two of its points can be separated by a countable point set.

real curve

A continuum is a *real curve* if it is a continuous 1-1 image of the real line.

recurrent point of

Let a continuum X and a mapping $f : X \to X$ be given. For each natural number n denote by f^n the n-th iteration of f. A point $p \in X$ is called a *recurrent point of* f provided that for every neighborhood U of p there is $n \in \mathbb{N}$ such that $f^n(p) \in U$. The set of recurrent points of a mapping $f : X \to X$ are denoted by $R(f)$.

refinable (monotonely)

A map r from a compact metric space X onto a compact metric space Y is *(monotonely) refinable* if, for each positive number ε, there is a (monotone) ε-map f from X onto Y such that, for each x in X,

$$d\left(f\left(x\right), r\left(x\right)\right) < \varepsilon.$$

regular

If X is a continuum and $p \in X$, then X is said to be *regular at* p provided that there is a local base $\mathcal{L}_p$ at p such that the boundary of each member of $\mathcal{L}_p$ is of finite cardinality. A continuum is said to be *regular* provided that X is regular at each of its points.

A continuum is regular if each two of its points can be separated by a finite point set.

retract, retraction

Let X and Y be continua. A mapping $f : X \to Y$ is said to be a *retraction* provided that $Y \subset X$ and the restriction $f|Y : Y \to f(Y) \subset X$ is the identity; then Y is called a *retract* of X.

rigid

A nondegenerate topological space X is said to be *rigid* if it has a trivial autohomeomorphism group, i.e., if the only homeomorphism of X onto X is the identity.

selectible

A continuum X is said to be *selectible* provided that there exists a mapping $\sigma : C(X) \to X$ (called a *selection* for $C(X)$) such that $\sigma(A) \in A$ for each continuum $A \subset X$ (see e.g. [Nadler 1978, p. 253]).

scattered

A point set X is *scattered* if every subset Y of X has a point that is not a limit point of Y.

self-homeomorphic

A topological space X is called *self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ such that V is homeomorphic to X.

semi-aposyndetic

A space X is *semi-aposyndetic* if, for each two of its points, X is aposyndetic at one of them with respect to the other.

semi-confluent

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]):
– *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

semi-continuum

A *semi-continuum* is a continuumwise connected point set.

semi-continuous

Let a continuum X, a compact space Y and a function $F : X \to 2^Y$ be given. Put

$$F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

The function F is said to be *lower (upper) semi-continuous* provided that $F^{-1}(B)$ is open (closed) for each open (closed) subset $B \subset Y$. It is said to be *continuous* provided that it is both lower and upper semi-continuous. This notion of continuity agrees with the one for mappings between metric spaces.

semi-locally connected

A continuum M is *semi-locally connected* at the point p if, for every domain D containing p, there exits a domain E lying in D and containing p such that $M \setminus E$ has only a finite number of components. A continuum is *totally non-semi-locally connected* if it is not semi-locally connected at any point.

simple closed curve

A *simple closed curve* is any space which is homeomorphic to the unit circle.

simple $\mathfrak{m}$-od

Let $\mathfrak{m}$ be a cardinal number. By a *simple* $\mathfrak{m}$*-od* with the center p we mean the union of $\mathfrak{m}$ arcs every two of which have p as the only common point.

simple triod

The union of three arcs emanating from a point v is called a *simple triod* provided the singleton v is the intersection of any two of the arcs ; the point v is then called the *top* of the simple triod, and the arcs are called its *arms*.

shore point

A point p of a dendroid X is called a *shore point* of X if there exists a sequence of subdendroids X_n of $X \setminus \{p\}$ such that $X = \operatorname{Lim} X_n$. A shore point of X that is not an end point of X is called an *improper shore point* of X (see [Neumann-Lara et al. 1993, p. 939]).

smooth (dendroid)

A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each point $x \in X$ and for each sequence of points $\{x_n\}$ in X tending to x, the sequence of arcs px_n tends to the arc px. A dendroid X is said to be *smooth* provided that it is smooth at some point $p \in X$ (see [Charatonik et al. 1970, p. 298]).

strongly chaotic

A nondegenerate topological space X is said to be *strongly chaotic* if for any two distinct points p and q of X there exist open neighborhoods U of p and V of q respectively such that no open subset of U is homeomorphic to any subset of V;

strongly pointwise self-homeomorphic

A topological space X is called *strongly pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a neighborhood V of x such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *strongly pointwise self-homeomorphic* if it is strongly pointwise self-homeomorphic at each of its points.

strongly rigid

A nondegenerate topological space X is said to be *strongly rigid* if the only homeomorphism of X into X is the identity of X onto itself.

strongly self-homeomorphic

A topological space X is called *strongly self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ with nonempty interior such that V is homeomorphic to X.

strongly unicoherent1

A continuum X is *strongly unicoherent* provided X is unicoherent and each proper subcontinuum with interior is unicoherent.

strongly unicoherent2

A continuum X is said to be *strongly unicoherent2* provided that it is unicoherent and for each pair of its proper subcontinua A and B such that $X = A \cup B$, each of A and B is unicoherent (see [Bennett 1971, p. 587]).

Suslinean

A continuum is *Suslinean* if it does not contain uncountably many mutually exclusive nondegenerate subcontinua.

$T_A(x)$

For each point x of a continuum X with an arc-structure A we define (see [Fugate et al. 1981, I.3, p. 550]) $T_A(x) = \{y \in X : \text{each convex subcontinuum of } X \text{ with } y \text{ in its interior contains } x\}$. Since $T_A(x)$ is always closed, we have $T_A : X \to 2^X$.

terminal continuum

A proper subcontinuum K of a continuum X is said to be a *terminal continuum* of X provided that if whenever A and B are proper subcontinua of X having union equal to X such that $A \cap K \neq \emptyset \neq B \cap K$, then either $X = A \cup K$ or $X = B \cup K$.

thin

A topological space S is *thin* if and only if for each two homeomorphic subset A and B of S, there is a homeomorphism h of S onto itself such that $h(A) = B$.

translation

If P and Q are subspaces of a metric space X with a metric d, and ε is a positive number, then a mapping $g : P \to Q$ is called an *ε-translation* provided that $d(p, g(p)) < \varepsilon$ for each point $p \in P$.

tree

A *tree* is a graph which contains no simple closed curve. A tree as a one-dimensional acyclic connected polyhedron, i.e., a dendrite with finitely many end points.

triod

A continuum is called a *triod* provided it contains a subcontinuum whose complement is the union of three nonempty pairwise disjoint open sets.

unicoherent

A connected topological space S is said to be *unicoherent* provided that whenever A and B are closed, connected subsets of S such that $S = A \cup B$, then $A \cap B$ is connected.

Let a continuum X and its subcontinuum Y be given. Then X is said to be *unicoherent at* Y provided that for each pair of proper subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B \cap Y$ is connected (see [Owens 1986, p. 146]).

uniformly continuum-chainable

A continuum X is said to be *uniformly continuum-chainable* if for each positive number ε there is an integer $k = k(\varepsilon)$ such that for each pair x,y of points of X, there are subcontinua $A_1, \cdots, A_k$ of X each of diameter less than ε such that $x \in A_1$, $y \in A_k$ and $A_i \cap A_j \neq \emptyset$ whenewer $|i - j| \leq 1$.

uniquely arcwise connected

A continuum is said to be *uniquely arcwise connected* provided that for every two of its points there is exactly one arc in the continuum joining these points.

universal

Let a class $\mathcal{S}$ of spaces be given. A member U of $\mathcal{S}$ is said to be *universal for* $\mathcal{S}$ if every member of $\mathcal{S}$ can be embedded in U, i.e., if for every $X \in \mathcal{S}$ there exists a homeomorphism $h : X \to h(X) \subset U$. Accordingly, a dendrite is said to be *universal* if it contains a homeomorphic image of any other dendrite.

weak cut point

A point x of a continuum M is a *weak cut point* of M if there are two points p and q of M such that every subcontinuum of M that contains both p and q also contains x.

weakly chainable

A continuum is *weakly chainable* if it is a continuous image of a chainable continuum.

weakly confluent

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]) *weakly confluent* provided that for each subcontinuum Q of Y there is a component the set $f^{-1}(Q)$ which is mapped under f onto Q.

weakly hereditarily unicoherent

A continuum is said to be *weakly hereditarily unicoherent* if the intersection of any two of its subcontinua with nonempty interiors is connected (see [Owens 1986, p. 152] and references therein).

weakly monotone

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [Maćkowiak 1979, Chapter 3 and 4, p. 12-28]) *weakly monotone* provided that for each subcontinuum Q of Y with the nonempty interior each component the set $f^{-1}(Q)$ is mapped under f onto Q.

weakly smooth at a point

A dendroid X is said to be *weakly smooth at a point* $p \in X$ provided that the subspace of 2^X consisting of all subarcs of X of the form px for $x \in X$ is compact (see [Lum 1975, p. 113]).

weakly Suslinian

A continuum X is said to be *weakly Suslinian* provided that X is not the union of more than one pairwise disjoint nondegenerate subcontinua of X.

widely connected

A nondegenerate connected space X is *widely connected* if each nondegenerate connected subset of X is dense in X.

width $w(X)$

For any compact metric space X, the *width* $w(X)$ of X is the l.u.b. of the set of all real numbers a which satisfy the following condition: for each $\varepsilon > 0$, there exists a finite open cover C of X such that $meshC) < \varepsilon$ and for each chain C' which is a subcollection of C there is a member A of C such that $d(A, C'^*) \geq a$.

Appendix B

List of examples

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