Examples in Continuum Theory

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Last section contains lost continua.

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The arc-like continua are beautiful

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The arc-like continua are beautiful

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The arc-like continua are beautiful

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Last section contains lost continua.

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The arc-like continua are beautiful

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Last section contains lost continua.

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The *Sierpiński Universal Plane Curve* is a well known continuum which serves as the universal element in the class of all one-dimensional continua in the plane. It is obtained as the residual set remaining when one begins with a square and applies the operation of dividing it into nine equal squares and omitting the interior of the center one, then repeats this operation on each of the surviving 8 squares, then repeats again on the surviving 64 squares, and so on $\cdots$. See [65, p.9].

The Sierpiński Universal Plane Curve can be characterized as the only plane locally connected one-dimensional continuum S such that the boundary of each

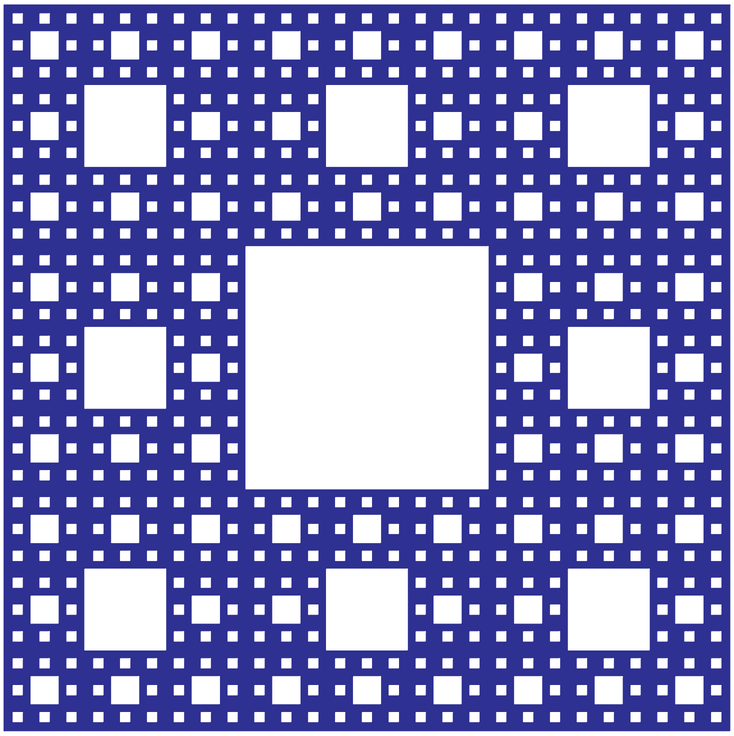


Figure 3.3.2: (A) Sierpiński Universal Plane Curve

complementary domain of S is a simple closed curve and no two of these complementary domain boundaries intersect. See [79].

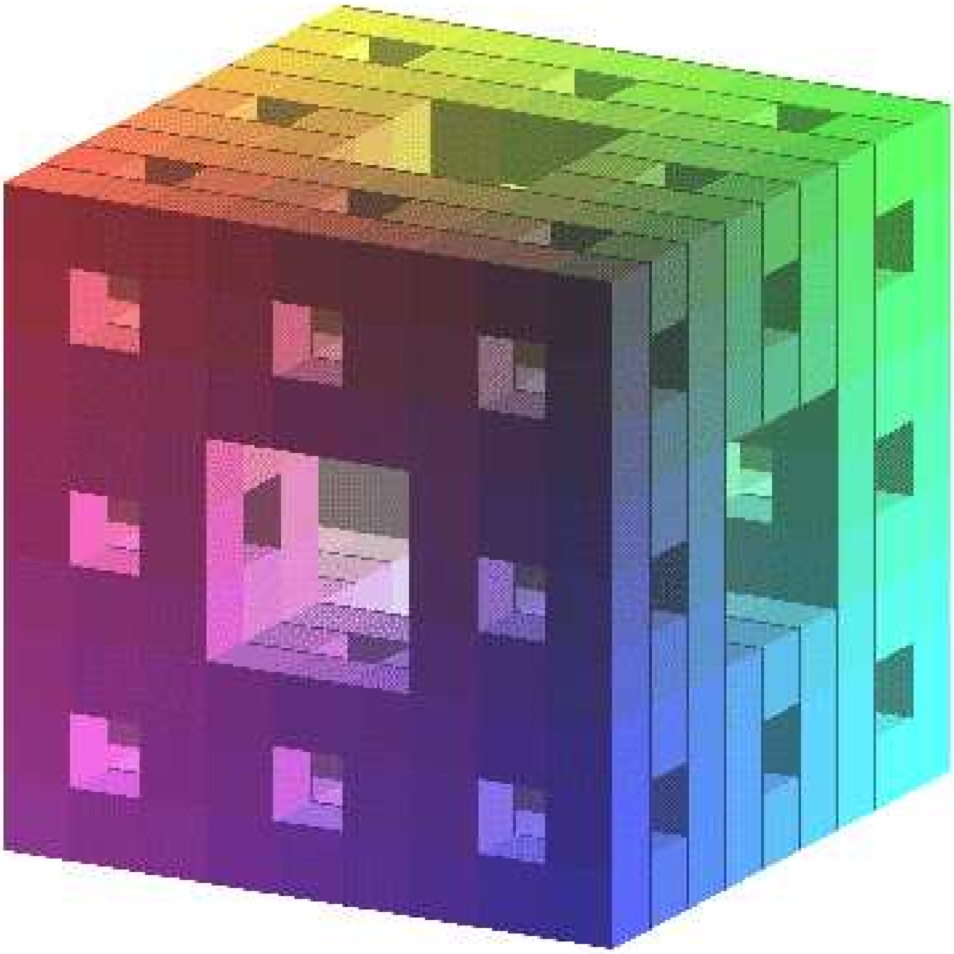
3.3.3 Menger's universal curve. Its Anderson's characterization.

The *Menger Universal Curve* X is obtained using the Sierpiński Universal Plane Curve S from the unit cube I^3 by the formula

$$X = \{(x, y, z) \in I^3 : (x, y) \in S \quad \& \quad (x, z) \in S \quad \& \quad (y, z) \in S\} \quad .$$

3.4 Universal Menger's compacta. Their Bestvina's characterization.

The arc-like continua are beautiful



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Last section contains lost continua.

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The arc-like continua are beautiful

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4.3.1 Sierpiński Tringle

The *Sierpiński Triangle* is obtained as the residual set remaining when one begins with a triangle and applies the operation of dividing it into four equal triangles and omitting the interior of the center one, then repeats this operation on each of the surviving 3 triangles, then repeats again on the surviving 9 triangles, and so on $\cdots$. See [36, Example 2.7].

The Sierpiński Triangle is homeomorphic to the unique nonempty compact set K of the complex plane that satisfies

$$K = w_1(K) \cup w_2(K) \cup w_3(K) \quad ,$$

where w_1, w_2, w_3 are maps of the complex plane defined by $w_1(z) = z/2$, $w_2 = (z+1)/2$ and $w_3 = (z+i)/2$.

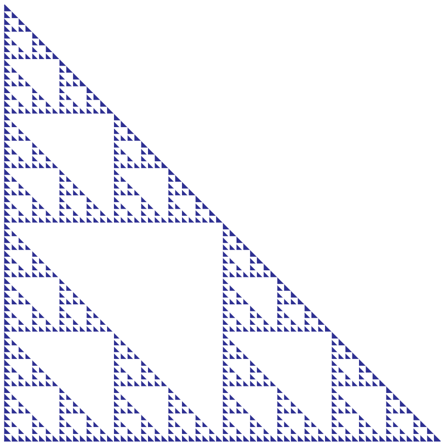

Figure 4.3.1: (A) Sierpiński Tringle

4.4 Urysohn's examples: for $\mathfrak{r} \in \{\omega, \aleph_0, \mathfrak{c}\}$ there is a continuum $X(\mathfrak{r})$ consisting solely of points of order $\mathfrak{r}$

The arc-like continua are beautiful

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Last section contains lost continua.

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The arc-like continua are beautiful

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Last section contains lost continua.

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Last section contains lost continua.

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Last section contains lost continua.

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Last section contains lost continua.

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Last section contains lost continua.

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Last section contains lost continua.

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Last section contains lost continua.

12.1 Hyperspaces are terrible ;-)

The arc-like continua are beautiful

Chapter 13

Lost continua

This section contains continua from the the first version.

13.1 Arc-like continua

The arc-like continua are beautiful

13.1.1 Topologist's Sin Curve

The *Topologist's Sin Curve* is well known continuum which is not arcwise connected. The continuum is defined by the union

$$\{(x, y) \in R^2 : 0 < x \leq 1, y = \sin(1/x)\} \cup \{(0, y) \in R^2 : -1 \leq y \leq 1\} \quad .$$

It has just two path components. More we can find in [74, p.137] .

13.1.2 Open Sin

The *Open Sin* is defined by the Euclidean closure of the set

$$\{(x, y) \in R^2 : -1 < x < 1, y = \sin(1/(x^2 - 1))\} \quad .$$

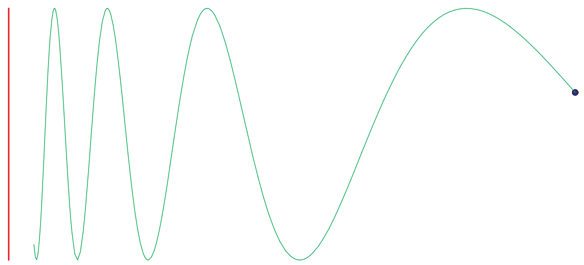


Figure 13.1.1: (A) Topologist's Sin Curve

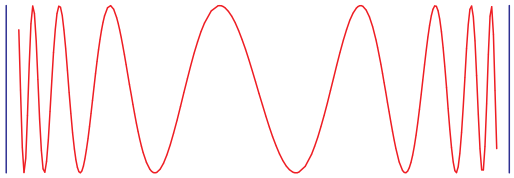


Figure 13.1.2: (A) Open Sin

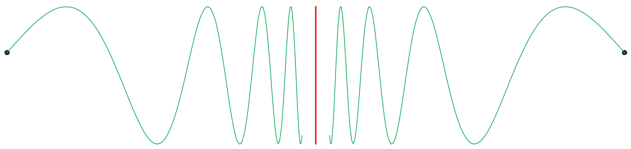


Figure 13.1.3: (A) Double Topologists's Sin Curve

This continuum has just three path components.

13.1.3 Double Topologists's Sin Curve

The *Double Topologists's Sin Curve* is defined by the union

$$\{(x,y) \in R^2 : 0 < |x| \leq 1, y = \sin(1/x)\} \cup \{(0,y) \in R^2 : -1 \leq y \leq 1\} \quad .$$

It has just three path components.

13.1.4 Snake

The *Snake* is an arclike continuum where each open mapping is a homeomorphism.

We can find more in [40, Theorem ..].

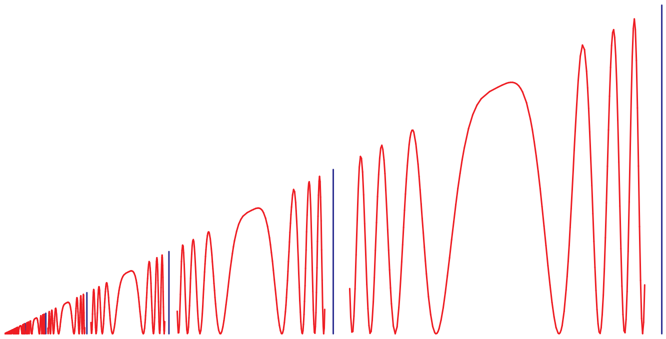


Figure 13.1.4: (A) Snake

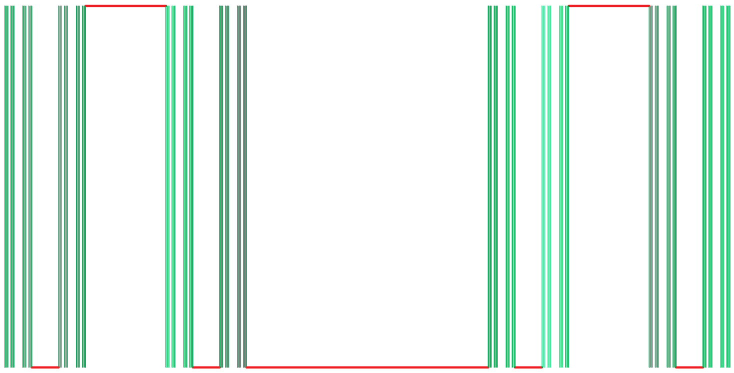


Figure 13.1.5: (A) Cantor Meanders

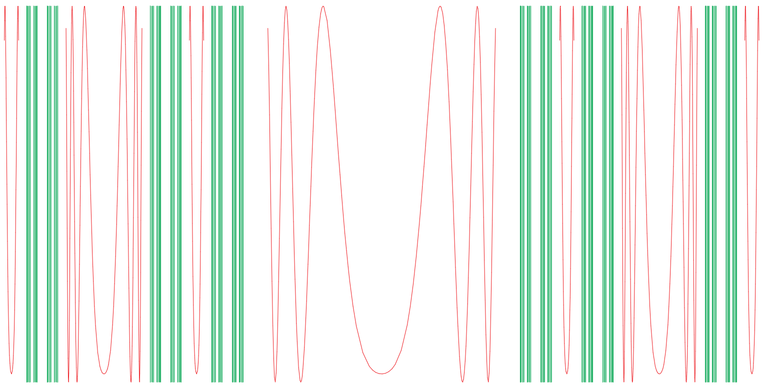


Figure 13.1.6: (A) Cantor Snake

13.1.5 Cantor Meanders

The *Cantor Meanders* X is formed by combining the product $C \times I$ of the Cantor Ternary Set C and the unit interval I with the appropriate segments of $I \times \{0, 1\}$. We use the 'zig-zag' system of selecting the intervals. We start with $[1/3, 2/3] \times \{0\}$, continue with $[1/9, 2/9] \times \{1\}$ and $[7/9, 8/9] \times \{1\}$, $\cdots$.

The space X has uncountably many arc components.

13.1.6 Cantor Snake

The *Cantor Snake* X is formed by combining the product $C \times I$ of the Cantor Ternary Set C and the unit interval I with countably many Open Sin Curves in such a way that the space X has uncountably many arc components.

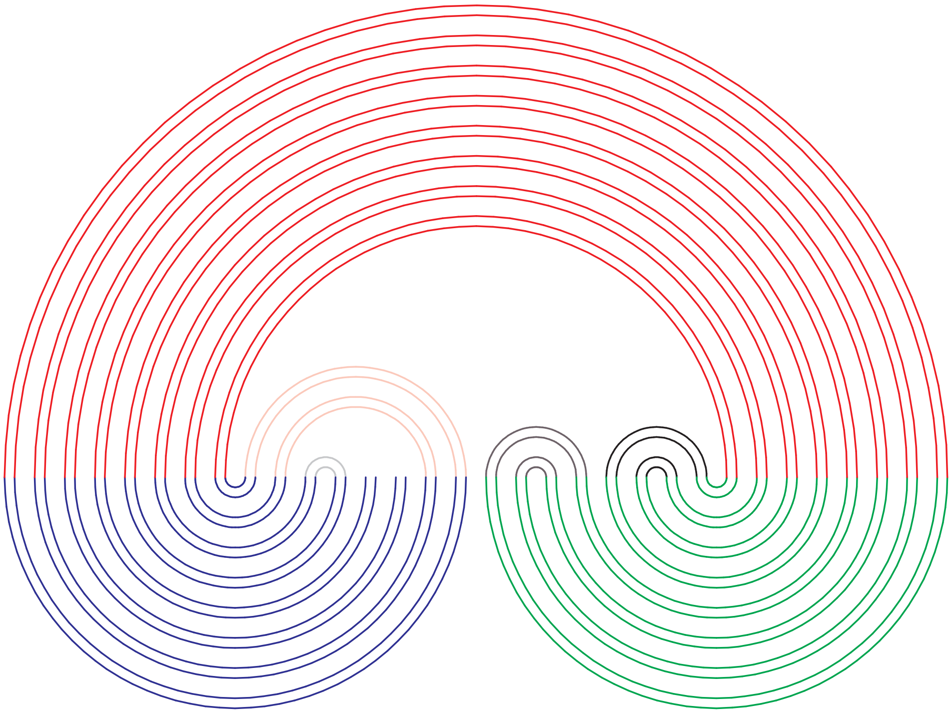


Figure 13.1.7: (A) Cruller

13.1.7 Cruller

The *Cruller* continuum is beautiful.

13.1.8 Double Buckethandle

The *Cantor* 1/5-*discontinuum* C_5 is created using the procedure of deleting the second and the fourth segments from five equal segments (instead of the middle third for the Cantor Ternary Set). C_5 is the set of the reals in the unit interval which can be represented without 1-s and 3-s in the form

$$\sum_{n=1}^{\infty} \frac{a_n}{5^n} \quad .$$

The *Double Buckethandle* continuum is the union of:

(i) the half circles in the lower plane going through points of $C_5 \cap \{x : 2/5^{n+1} \leq x \leq 1/5^n\}$ with the centre $7/10 \cdot 5^n$,

(ii) the half circles in the upper plane going through points of $C_5 \cap \{x : 2/5^{n+1} \leq 1 - x \leq 1/5^n\}$ with the centre $1 - 7/10 \cdot 5^n$.

We can find more in [55, p.205 - p.206].

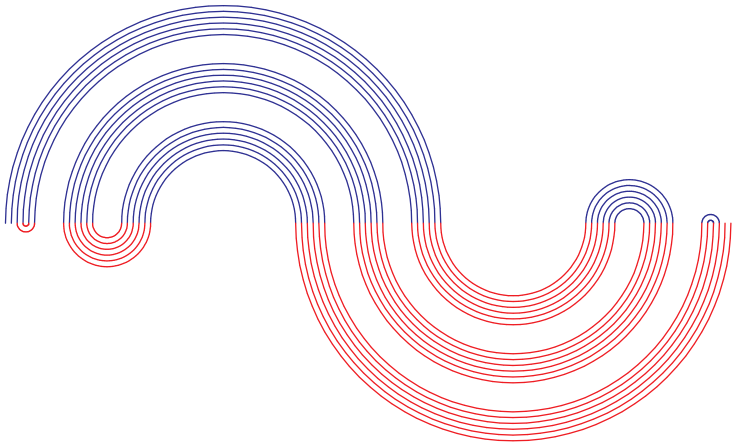


Figure 13.1.8: (A) Double Buckethandle

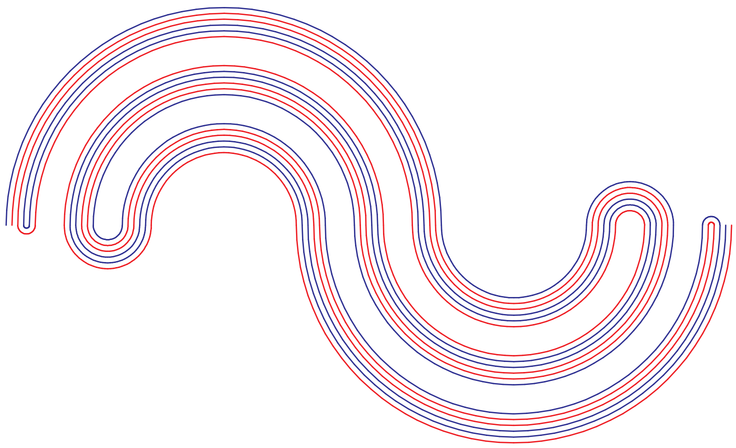


Figure 13.1.8: (B) Double Buckethandle

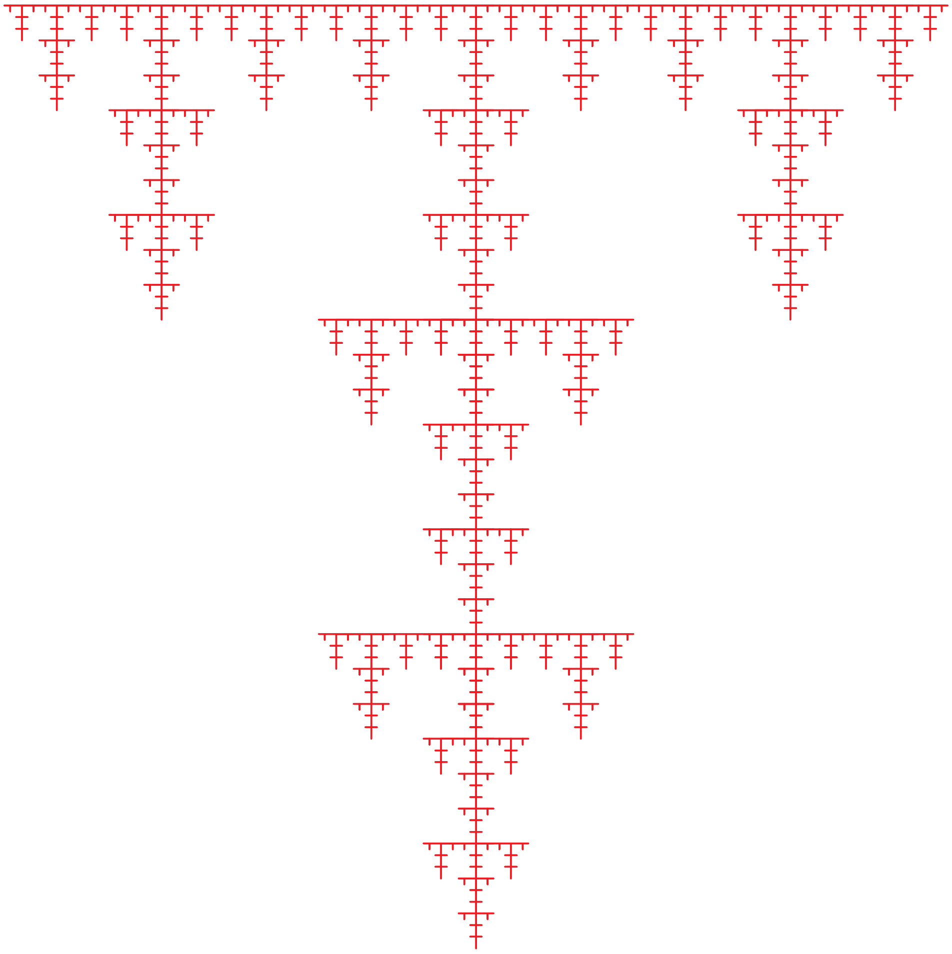


Figure 13.2.1: (A) Root

13.2 Self-homeomorphic continua

Self-homeomorphic continua are beautiful

13.2.1 Root

The *Root* is the continuum obtained attaching small 'T' letters to the previous ones as in the figure. It is a strongly pointwise self-homeomorphic dendrite. It has points of order three on the horizontal segments and points of order four on the vertical segments.

See [36, p.232-233].

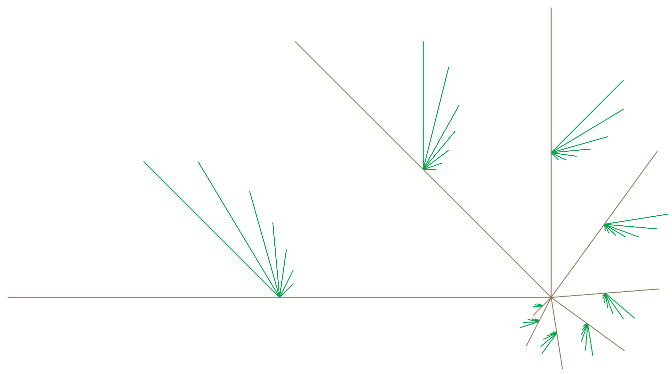


Figure 13.3.1: (A) construction of the Ważewski's Universal Dendrite

13.3 Dendrites

Dendrites are beautiful

13.3.1 Ważewski's Universal Dendrite

The *Ważewski's Universal Dendrite* is well known continuum which serves as the universal element in the class of all dendrites. We start with a Locally Connected Fan F_ω. Denote $D_1 = F_\omega$. Now we attach sufficiently small copies of F_ω to all midpoints of all maximal free arcs in D_1 and obtain a dendrite D_2. Continuing in this fashion, we obtain the sequence of dendrites $\{D_n\}$ and the Ważewski's Universal Dendrite $\mathbf{D}$ is the closure of their union. For details see [65, p.181-185].

13.3.2 Universal Dendrite D_3

The *Universal Dendrite* D_3 is obtained by the same procedure as the Ważewski's Universal Dendrite just instead of locally connected fans we use simple triods.

We start with a simple triod T. Denote $T_1 = T$. Now we attach sufficiently small copies of T (with one end point as the attaching point) to all midpoints of all maximal free arcs in T_1 and obtain a dendrite T_2. Continuing in this fashion, we obtain the sequence of dendrites $\{T_n\}$ and the Universal Dendrite $\mathbb{D}_3$ is the closure of their union.

The Universal Dendrite $\mathbf{D_3}$ is homogeneous with respect to the monotone mappings. For details see [38, p.362].

13.3.3 Universal Dendrite D_4

The *Universal Dendrite* D_4 is a dendtrite with ramification points of order 4.

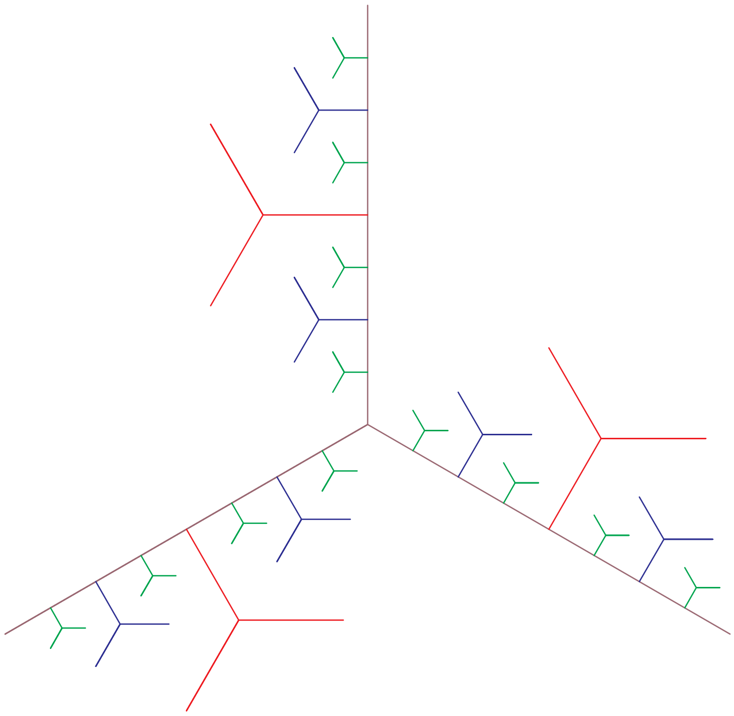


Figure 13.3.2: (A) Universal Dendrite D_3

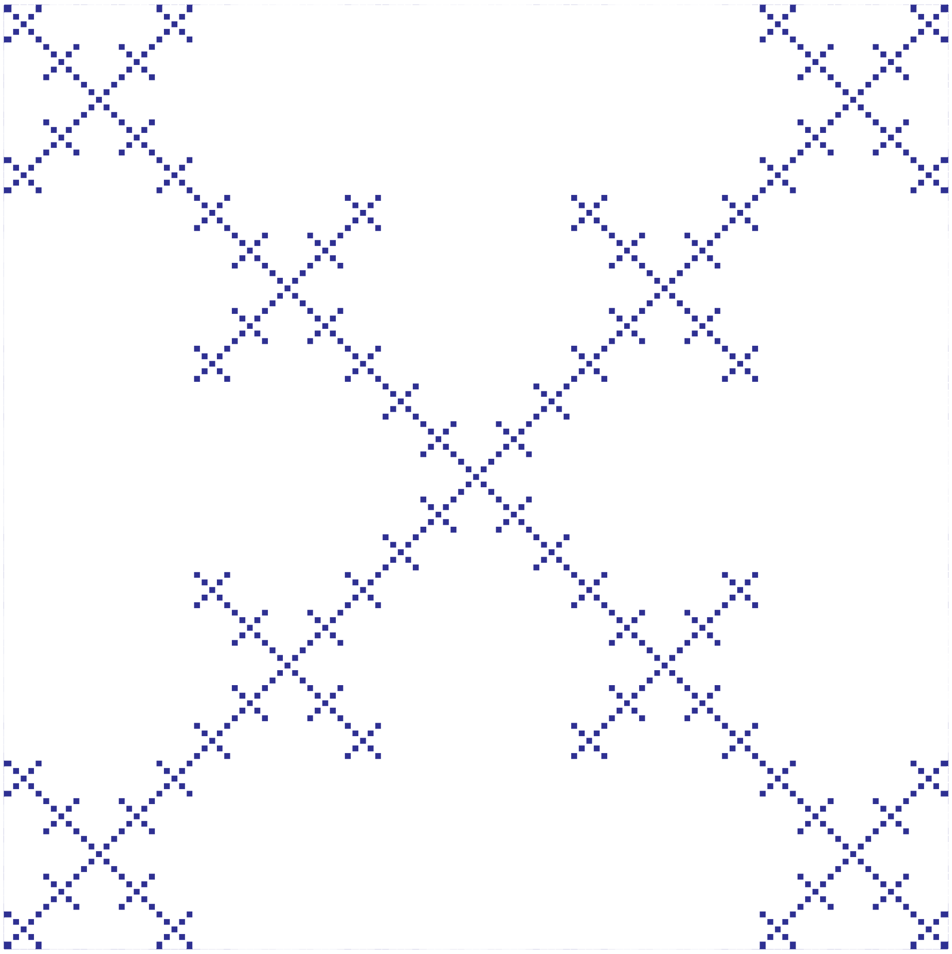


Figure 13.3.3: (A) Universal Dendrite D_4

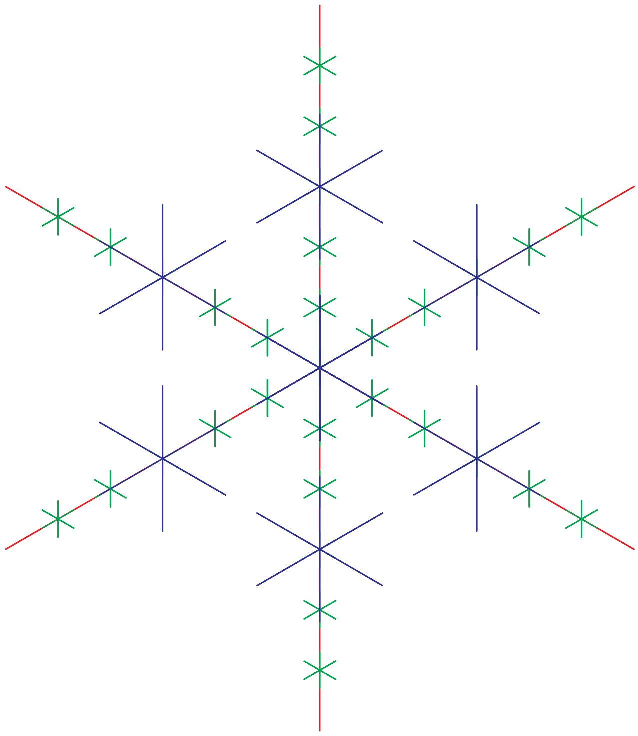


Figure 13.3.4: (A) Universal Dendrite D_6

13.3.4 Universal Dendrite D_6

Universal Dendrite D_6 is nice.

13.3.5 Gehman Dendrite

The *Gehman Dendrite* G is formed from the Cantor Ternary Set C in such a way that C is the set of end points of G and all ramification points of G are of order three.

See [67, pp. 422-423] and [68, p. 203].

13.4 Dendroids

Dendroids are beautiful

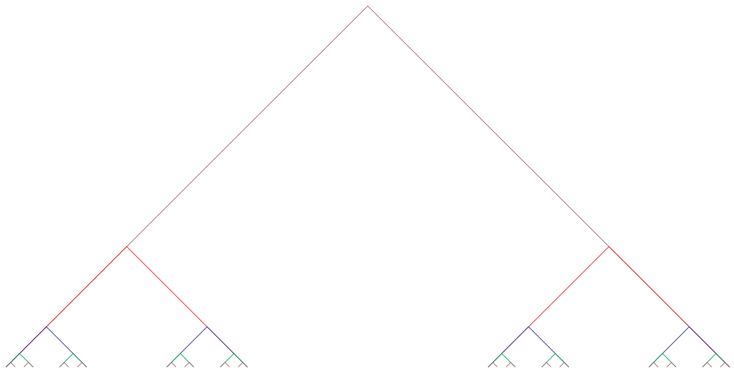


Figure 13.3.5: (A) Gehman Dendrite

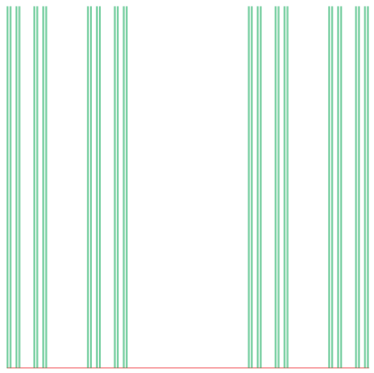


Figure 13.4.1: (A) Cantor Comb

13.4.1 Cantor Comb

The *Cantor Comb* X is formed from the Cantor Ternary Set C and the unit interval I using the formula

$$X = C \times I \quad \cup \quad I \times \{0\} \quad .$$

13.4.2 Gehman Dendrite with Ski

The *Gehman Dendrite with Ski* G_s is a dendroid formed from the Gehman Dendrite G and the Cantor Ternary Set C with a formula

$$G_s = \{(x, y, z) \in R^3 : (x, y) \in G\} \cup \{(x, y, z) \in R^3 : x \in C, y = 0, |z| \leq 1\} \quad .$$

In fact we attach uncountably many segments of length 2 with their midpoints to the points of Cantor Ternary Set (the end points of Gehman Dendrite). See [25, p. 16].

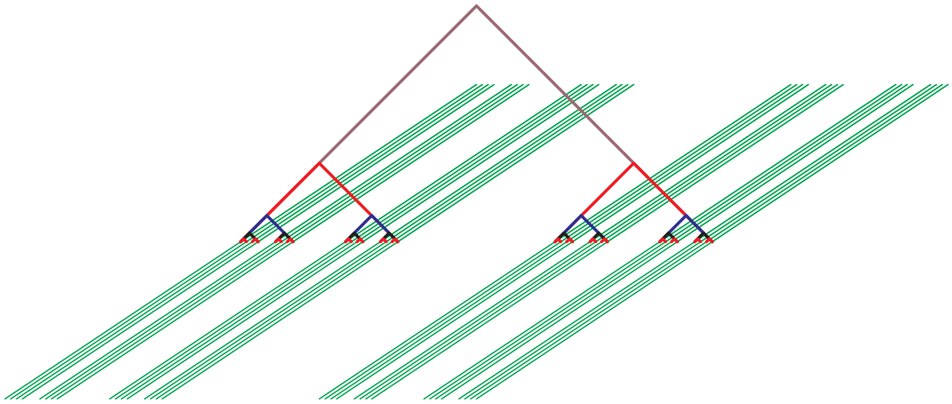


Figure 13.4.2: (A) Gehman Dendrite with Ski

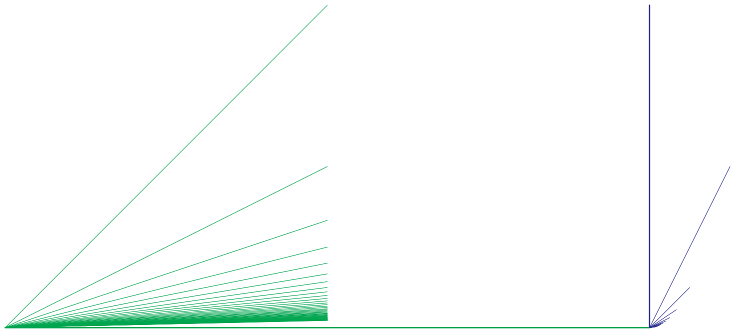


Figure 13.4.3: (A) Makuchowski Twin

13.4.3 Makuchowski Twin

We obtain the *Makuchowski Twin* as a subset of the Euclidean plane with the induced topology by attaching the Locally Connected Fan to the end point of the prolonged segment of the Prolonged Harmonic Fan.

Makuchowski twin is a dendroid where each nonconstant open mapping is light. See [60, Example 2.1].

13.5 Fans

Fans are beautiful

13.5.1 Harmonic Fan

We define the *Harmonic Fan* $F_H = \cup\{ve : e \in H\}$, where ve is the segment joining $v = (0, 1)$ with $e \in H$, $H = \{0\} \cup \{1/n : n \in N\}$. We consider F_H as a

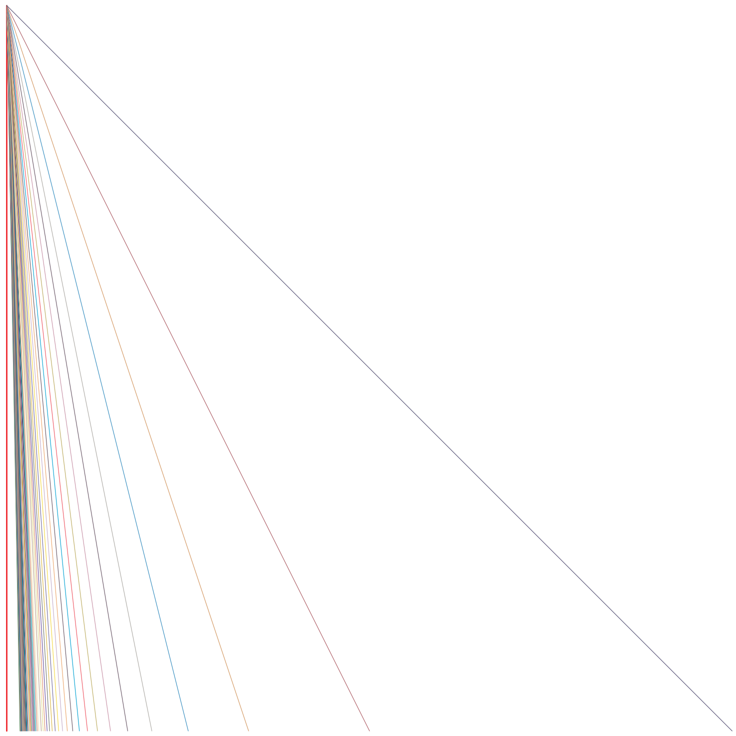


Figure 13.5.1: (A) Harmonic Fan

subset of the Euclidean plane with the induced topology.

See [39, p.9].

13.5.2 Harmonic Shredded Fan

To define the Harmonic Shredded Fan F_{HS} we cut the 'arms' ve in the Harmonic Fan F_H in such a way, that their lengths run through all rationals in the unit interval (we do not cut the vertical segment through the origin). We notice that all Harmonic shredded fans are homeomorphic.

See [39, p.31].

13.5.3 Cantor Fan

We define the *Cantor Fan* $F_C = \cup\{ve : e \in C\}$, where ve is the segment joining $v = (1/2, 1)$ with all points e in the Cantor Ternary Set C. We consider F_C as a subset of the Euclidean plane with the induced topology.

See [39, p.9].

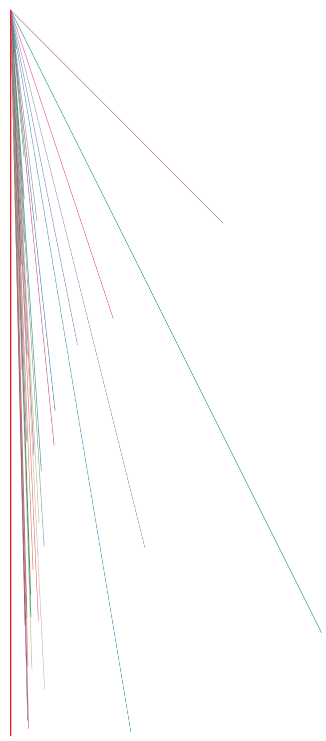


Figure 13.5.2: (A) Harmonic Shredded Fan

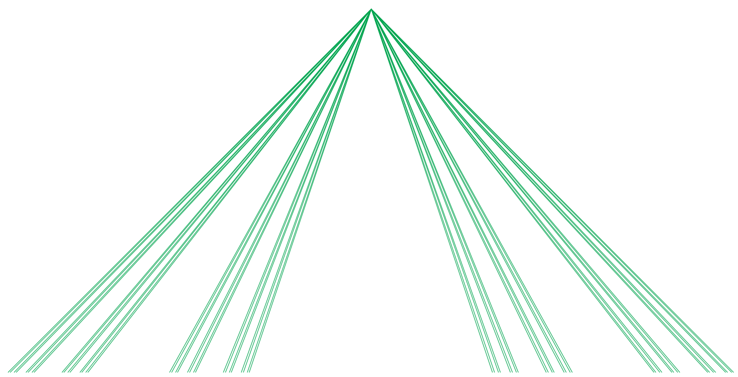


Figure 13.5.3: (A) Cantor Fan

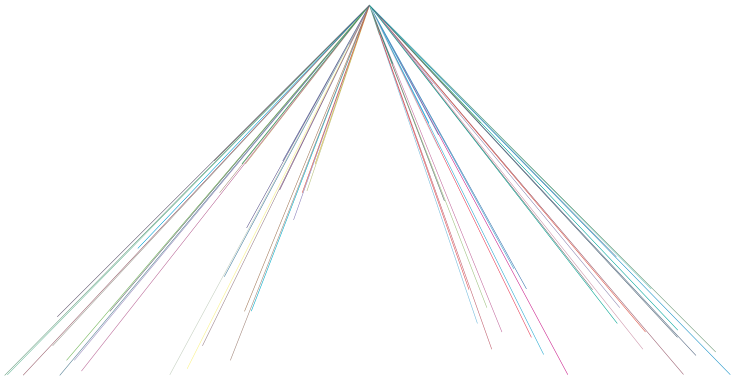


Figure 13.5.4: (A) Lelek Fan

13.5.4 Lelek Fan

We define the *Lelek Fan* F_L a subfan of the Cantor Fan F_C such that the set $E(F_L)$ of its end points is dense in F_L, and $S(F_L) = \{v\} \cup E(F_L)$ is a connected set (where v denotes the top of the two fans F_C and $F_L \subset F_C$).

It is known, that any two Lelek Fans are homeomorphic. See [39, p.9] . Moreover the Lelek Fan is homeomorphic to all its confluent images (see [37,]).

13.6 Planable continua

Planable continua are beautiful

13.6.1 Makuchowski Umbrella

Makuchowski Umbrella is a hereditarily locally connected continuum where each nonconstant open mapping is light. This property disappears when we cut out those vertical segments due to the horizontal projection on appropriate vertical segment.

See [60, Example 2.5].

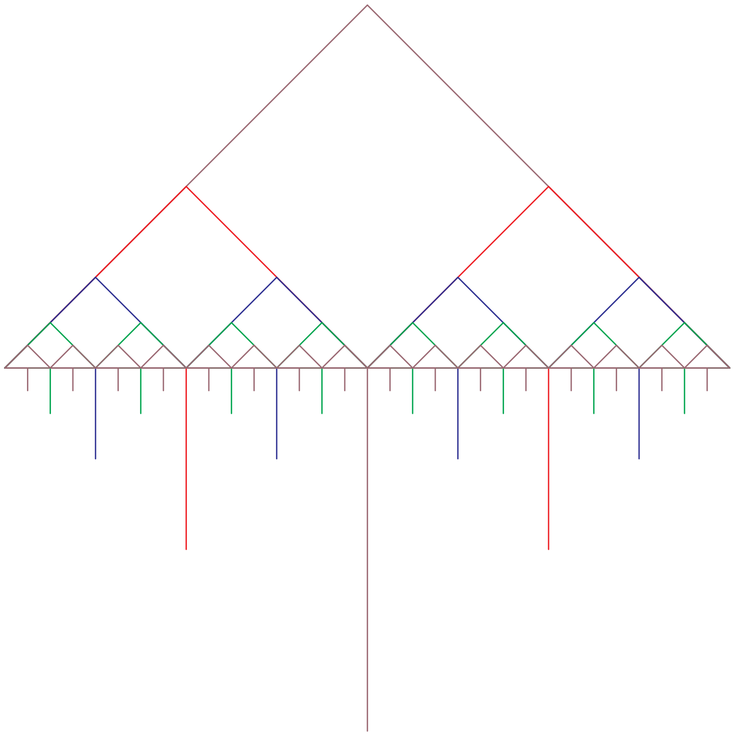


Figure 13.6.1: (A) Makuchowski Umbrella

Chapter 14

New examples

New section contains new examples.

14.1 New examples

New examples are nice.

14.1.1 Whyburn's Curve

The well-known Jordan Curve Theorem says that a simple closed curve in the plane cuts the plane into two regions and is their common boundary. Thus, it is a cutting of the plane. The so called Warsaw Circle, obtained from the $\sin(1/x)$-curve

$$S = \{(0,y) \in \mathbb{R}^2 : y \in [-1,1]\} \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 : x \in (0,1]\}$$

by identifying the points $(0,-1)$ and $(1, \sin 1)$ also is a cutting. But both these continua contain arcs, which certainly do not cut the plane. So, one can ask a question if there exists such a cutting $X \subset \mathbb{R}^2$ that each nondegenerate subcontinuum of X also cuts $\mathbb{R}^2$. The question has been answered in the affirmative in 1930 by G. T. Whyburn who constructed in [3] a *hereditary cutting of the plane*, which is now called the *Whyburn's Curve*. The present description of this peculiar continuum is a modification of the original one due to A. Lelek [1], to obtain some extra properties of the curve, and it follows Lelek's sketch of the construction in [3, p. 117-119]. We say that a continuum X lying in the plane $\mathbb{R}^2$ cuts the plane between points a and b if the points belong to different components of $\mathbb{R}^2 \setminus X$. A continuum X is a cutting of $\mathbb{R}^2$ if there are some two points such that X cuts $\mathbb{R}^2$ between them. Take the $\sin(1/x)$curve S and identify points $(0,1)$ and $(0,-1)$ into one point r. The obtained curve (still considered as a subset of the plane) consists of a circle C and a sinusoidal curve approximating the circle, so it is a cutting of the plane. If we add to the curve an arc starting at r and having only this point in common with the curve, we get a continuum Y being also a cutting of the plane (namely it cuts the plane between a point a inside the circle C and a point b lying in $\mathbb{R}^2 \setminus Y$ outside C) and having the property that every subcontinuum of Y containing a point p of the sinusoidal part of Y and a point q lying on the added arc, also cuts the plane between a and b (see Figure A (I)-(II).).

The same curve Y can also be obtained as the intersection of a nested sequence of plane continua. Each continuum of the sequence has the form of a long strip with one loop and finitely many zig-zags. Consecutive strips are thinner and thinner, more and more narrow, each next being inscribed in the previous one in such a way that the next loop is contained in the previous loop which also contains one more zig-zag of the next strip (see Figure A (III)-(IV)). Consequently, the number of zig-zags in the next approximation is greater than that in the previous one, so these numbers tend to infinity. The common part of the all strips is just the curve Y, while the common part of the loops is the circle C contained in Y.

The Whyburn's curve W can be obtained in a similar way, as the intersection of a nested sequence of strips in the plane, using the method called "condensation of singularities". Namely the singularity appearing in each neighborhood of C in Y is copied more and more times both in the loops as well as in the zig-zag parts of the consecutive strips. A little bit more precisely speaking, the strips are modified so that each next strip, magnifying the number of zig-zags in the loops and preserving previous loops, contains also new, smaller loops, distributed in a more and more dense manner inside the previous strip (see Figure A (V)) so that the diameters of the loops tend to zero as the number of steps of the construction tends to infinity. As a consequence of this property, for every two points p and q of the curve W one can find, in a sufficiently far approximating strip, a loop lying between these points. Roughly speaking, behavior of the curve W between p and q is such as that of the curve Y, i.e., each subcontinuum of W containing p and q cuts the plane between some points a and b determined by the mentioned loop.

The continuum W constructed is this way has, among others, the following properties (see [1]). 1) each subcontinuum of W is a cutting of the plane, and therefore W contains no arc; 2) each subcontinuum of W contains a homeomorphic copy of W; 3) W is hereditarily decomposable; and 4) W is a continuous image of the pseudo-arc $\mathbb{P}$. Since neither the pseudo-arc $\mathbb{P}$ nor any subcontinuum of $\mathbb{P}$ cuts the plane, this last property shows that a hereditary anti-cutting of $\mathbb{R}^2$ can be continuously mapped onto a hereditary cutting.

[1] A. Lelek : On weakly chainable continua, Fund. Math. 51 (1962), 271–282.

[2] A. Lelek : Zbiory, PZWS, Warszawa 1966.

[3] G. T. Whyburn : A continuum every subcontinuum of which separates the plane, Amer. J. Math. 52 (1930), 319–330.

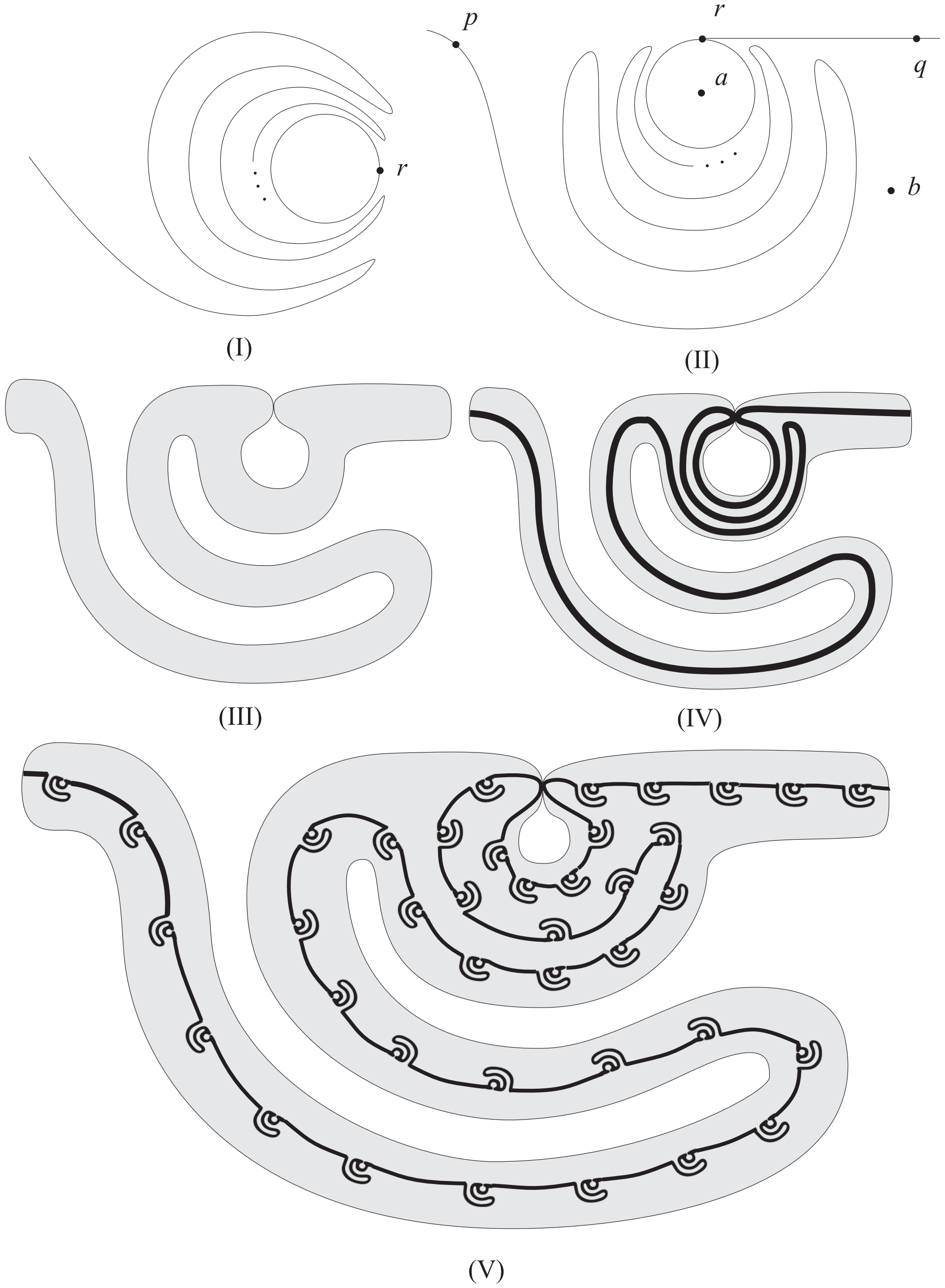


Figure 14.1.1: (A) construction of the Whyburn's Curve

14.1.2 Pseudo-arc

A. Lelek, ZBIORY (Sets), PZWS Warszawa 1966, pages 90-95.The first heredi-

tarily indecomposable continuum has been constructed by B. Knaster in 1922. This continuum K, discovered by Knaster, is just named the *pseudo-arc* or *K-naster's continuum*. The pseudo-arc K is obtained as the common part of a decreasing sequence of continua in the plane, every of which is the union of finitely many discs, forming a chain of sets. We will consider chains

$$\mathcal{D} = (D_1, D_2, \ldots, D_k)$$

of discs D_i in the plane, having the property that any two consecutive disc D_i and D_{i+1} have interior points in common, i.e., that their intersection — nonempty by assumption — is not contained in the boundary of any of them (Figure A). Discs D_i are called *links* of the chain $\mathcal{D}$. The union of all links of the chain $\mathcal{D}$ is a continuum. Observe that if we delete from the chain $\mathcal{D}$ a link which is not an end link, i.e., a link D_i such that $1 < i < k$, then the union of the remaining links is a not connected set, having two components.

Let two chains of discs $\mathcal{D}$ and $\mathcal{D}'$ be given. We say that the chain $\mathcal{D}'$ *refines* the chain $\mathcal{D}$ if each link D'_s of the chain $\mathcal{D}'$ is contained in the interior of at least one link D_i of the chain $\mathcal{D}$, i.e., if for every $D'_s \in \mathcal{D}'$ there exists a link $D_i \in \mathcal{D}$ such that $D'_s \subset D_i$ and that the disc D'_s has no common points with the boundary of the disc D_i (Figure B).

We say that the chain $\mathcal{D}'$ is *crooked* in the chain $\mathcal{D}$ if it refines $\mathcal{D}$ and if, for every pair of links D_i and D_j of the chain $\mathcal{D}$ such that

$$i + 2 < j, \tag{1}$$

and for every pair of links D'_s and D'_v of the chain $\mathcal{D}'$, intersecting the links D_i and D_j respectively, i.e.,

$$D_i \cap D'_s \neq \emptyset \neq D_j \cap D'_v,$$

there are, in the chain $\mathcal{D}'$, links D'_t and D'_u, lying between the link D'_s and D'_v in the same order, i.e.,

$$s < t < u < v \quad \text{or} \quad s > t > u > v,$$

such that

$$D'_t \subset D_{j-1} \quad \text{and} \quad D'_u \subset D_{i+1}. \tag{2}$$

In other words, a part of the chain $\mathcal{D}'$ which is contained between the links D'_s and D'_v has to form a fold in the chain $\mathcal{D}$ between the links D_i and D_j (Figure C). The chain $\mathcal{D}'$, to go from the link D_i to the link D_j has first to come to the link D_{j-1}, next go back to the link D_{i+1}, and only after this it can reach the link D_j. Such a fold must exist for every two links D_i and D_j having no adjoining (i.e. neighbor) link in common. Now, let there be given in the plane two points a and b, and an infinite sequence

$$\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n, \ldots$$

of chains of discs, satisfying, for every $n \in \{1, 2, \ldots\}$, the following conditions:

1^o the point a belongs to the first, and the point b to the last link of the chain $\mathcal{D}_n$;

2^o the diameter of every disc in the chain $\mathcal{D}_n$ is less than $\frac{1}{n}$;

3^o the chain $\mathcal{D}_{n+1}$ is crooked in the chain $\mathcal{D}_n$.

Thus, for example, the chain $\mathcal{D}_2$ is crooked in the chain $\mathcal{D}_1$, and the chain $\mathcal{D}_3$ is crooked in the chain $\mathcal{D}_2$ (on Figure D only a part of the chain $\mathcal{D}_3$ is presented). The foldings are more and more condensed and they overlapped themselves. Denote by K_n the union of all links of the chain D_n. From condition 3^o it follows in particular that the each next chain refines the previous one, and therefore the continua K_n form a decreasing sequence

$$K_1 \supset K_2 \supset \ldots \supset K_n \supset \ldots$$

The *Pseudo-arc* K is defined as the common part of all continua K_n, i.e.,

$$K = \bigcap \{K_n : n \in \mathbb{N}\}.$$

By virtue of condition 1^o points a and b belong to the continuum K, thus it is nondegenerate. We shall prove that the continuum K is hereditarily indecomposable.Let $K' \subset K$ be an arbitrary continuum. To prove that the continuum K is indecomposable it is enough to show that every continuum $Y \subset K'$ distinct from K' is nowhere dense in K'. Consider an arbitrary point $p \in Y$ and number $\varepsilon > 0$.

Since $K' \neq Y$, hence there is a point $q \in K' \setminus Y$ which therefore is at a distance at least $\eta > 0$ from each point $y \in Y$, where η is a fixed number, independent from y. In fact, if not, then some points of the set would be arbitrarily close to the point q, and so the point q would belong to the closure of the set Y, which is impossible, because the set Y is closed, and it does not contain the point q. Let us take a natural number n so large that the inequalities are satisfied

$$\frac{2}{n} < \varepsilon \quad \text{and} \quad \frac{3}{n} < \eta.$$

Denote by D_i and D_j (with $i \leq j$) the links of the chain $\mathcal{D}_n$ whose union contains the points p and q. Since $p \in Y$, hence $\rho(p, q) \geq \eta$, (here ρ means the metric in the plane) and thus it follows that between the links D_i and D_j there are at least two links of the chain $\mathcal{D}_n$. Indeed, in the opposite case the links D_i and D_j would have a neighbor common link, and choosing suitable points p' and q' in the intersections of this link with the links D_i and D_j we would have

$$\rho(p, q) \leq \rho(p, p') + \rho(p', q') + \rho(q', q) < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{3}{n} < \eta$$

by condition 2^o. Thus we can assume that the indices i and j satisfy the inequality (1). Further, we can assume also that the point p belongs to one, and the point q to the other of the links D_i and D_j.

Denote by D'_s and D'_v the links of the chain $\mathcal{D}_{n+1}$ which contain points p and q, respectively. Additionally we assume that $p \in D_i$. In the opposite case, i.e. if $p \in D_j$, the further part of the proof runs in the same way (with changing of the roles of the links D_i and D_j as well as D_{i+1} and D_{j-1}).

So,

$$p \in D_i \cap D'_s \quad \text{and} \quad q \in Dj \cap D'_v,$$

whence we infer, by virtue of condition 3^o and the definition of folding of chains, that there are two links D'_t and D'_u in the chain $\mathcal{D}_{n+1}$ which lie between the links D'_s and D'_v in the same order and which satisfy the inclusions (2). The union of the links of the chain $\mathcal{D}_{n+1}$ distinct from the link D'_u is a not connected set (see above) containing the point p in one component, and the point q in the other. Since the continuum K' joins the points p and q and is contained in the union K_{n+1} of all links of the chain $\mathcal{D}_{n+1}$, hence K' must pass thru the link D'_u. Thus there is a point

$$x \in K' \cap D'_u$$

(Figure E). We shall prove that the point x does not belong to the continuum Y. Indeed, if it would be $x \in Y$, then by the same reason as previously for the continuum K' the continuum Y would have to pass thru the link D'_t. Then there would exist a point

$$y \in Y \cap D'_t,$$

which would be in the link D_{j-1} according to (2). But the neighbor links D_j and D_{j-1} have their diameters less than $\frac{1}{n}$ by condition 2^o, and therefore we would have

$$\rho(y, q) < \frac{2}{n} < \frac{3}{n} < \eta,$$

which is impossible because $y \in Y$.

So, the point x belongs to the set $K' \setminus Y$ and to the link D_{i+1} simultaneously. The neighbor links D_i and D_{i+1} have diameters less than $\frac{1}{n}$, whence it follows that

$$\rho(p, x) < \frac{2}{n} < \varepsilon.$$

Since ε was an arbitrary positive number, hence, in this way we have proved in that there are points of the set $K' \setminus Y$ which lie arbitrarily closely to the point p. The point p was an arbitrary point of the continuum Y. Thus we have proved that the continuum Y is nowhere dense in K', and the proof of hereditary indecomposability of the continuum K is finished.

An example of a chain crooked in a chain with 7 links is on Figure F.

14.1.3 Dyadic Selenoid

The *Dyadic Selenoid* is obtained easily :-)

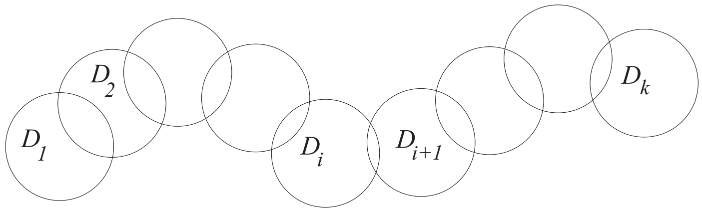


Figure 14.1.2: (A) a chain with links, construction of the Pseudo-arc

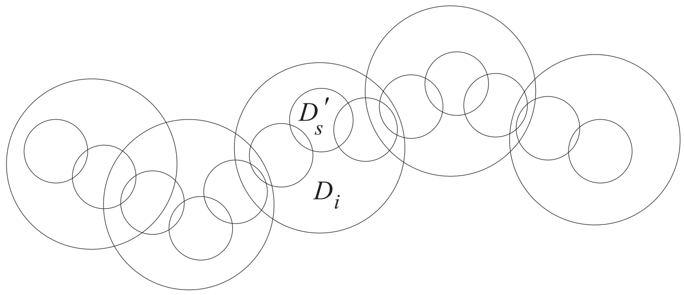


Figure 14.1.2: (B) refining a chain, construction of the Pseudo-arc

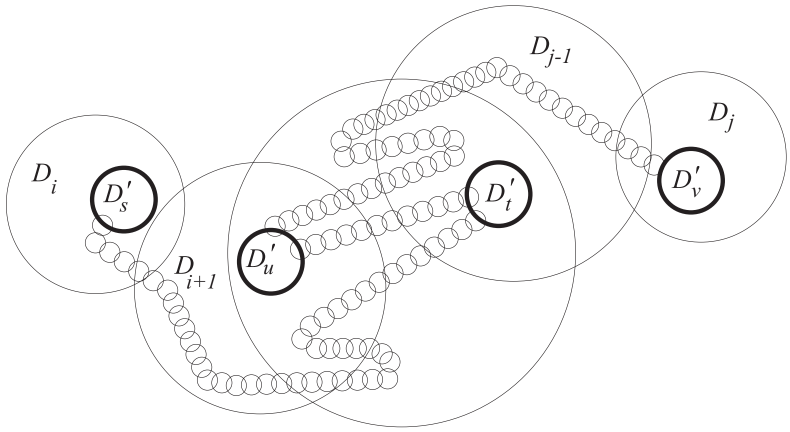


Figure 14.1.2: (C) crookedness used in the construction of the Pseudo-arc

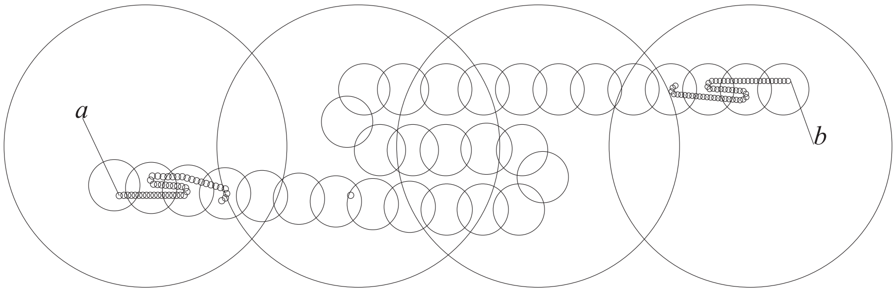


Figure 14.1.2: (D) crookedness used in the construction of the Pseudo-arc

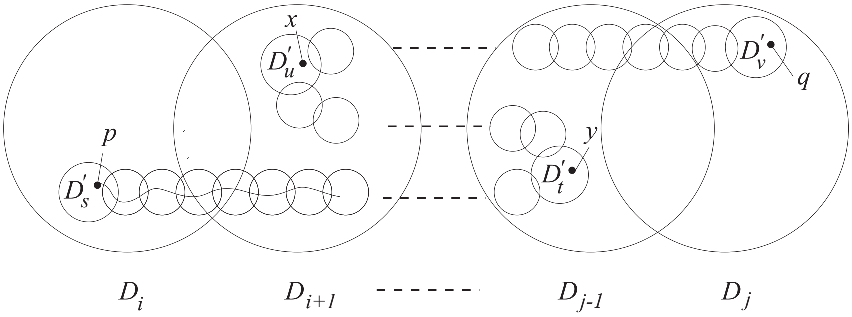


Figure 14.1.2: (E) crookedness used in the construction of the Pseudo-arc

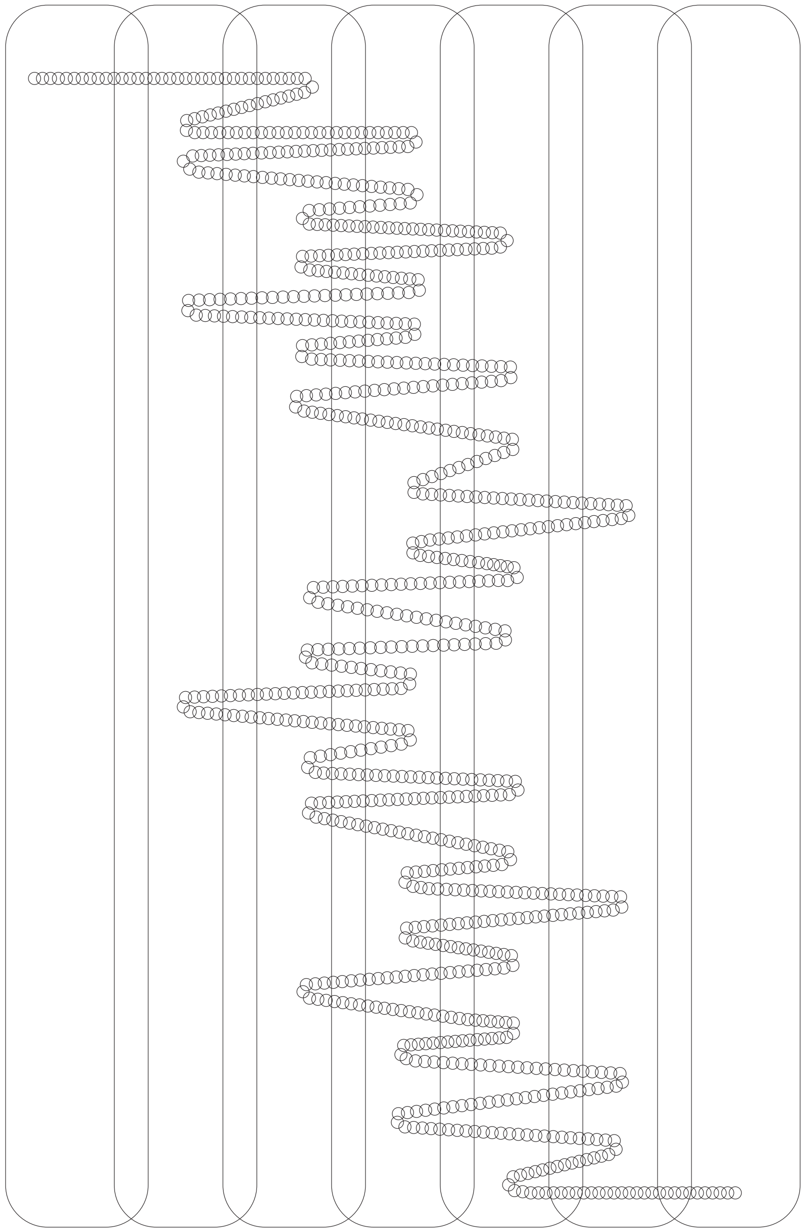


Figure 14.1.2: (F) an example of a chain crooked in a chain with 7 liks, construction of the Pseudo-arc

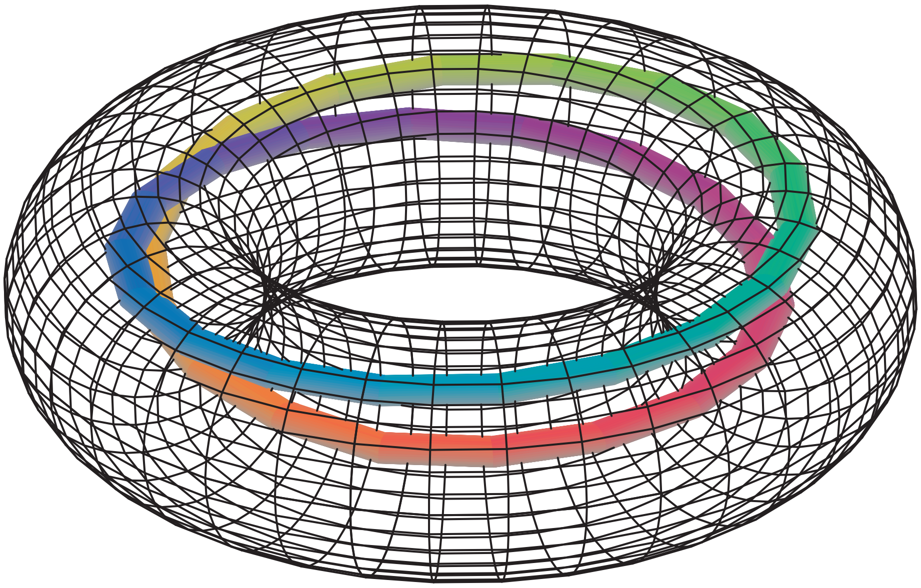


Figure 14.1.3: (A) first step in the construction of the Dyadic Selenoid

14.1.4 Zig-zag Circle

The *Zig-zag Circle* is obtained easily :-)

14.1.5 M-Continuum

The *M-Continuum* is obtained easily :-)

14.1.6 Buckethandle

The *Buckethandle* is created from the Cantor Ternary Set C with this procedure:

(i) we join any two points a and b in C symmetric with respect to $1/2$ with a semicircle in the upper half plane with the centre in $(1/2, 0)$,

(ii) we join any two points of C in the interval $2/3^n \leq x \leq 3/3^n$, $n \geq 1$, with a semicircle with the centre in $(5/(2 \cdot 3^n), 0)$ in the lower half plane.

This continuum is often called the *Knaster's Buckethandle* continuum (see Figure (A)).

This kind of curiosities in continuum theory is connected with common boundary of several plane regions. In 1904 A. Schönflies started to publish a sequence of papers [6] which became an important step in development of continuum theory. Relying heavily on intuition, he claimed that there do not exist three regions in the plane with common boundary. The claim was refused by L. E. J. Brouwer in 1910 [1] who constructed continua which are common boundary of three regions and showed that they are *indecomposable*, i.e., such

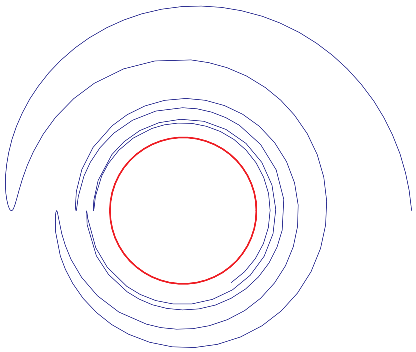


Figure 14.1.4: (A) Zig-zag Circle

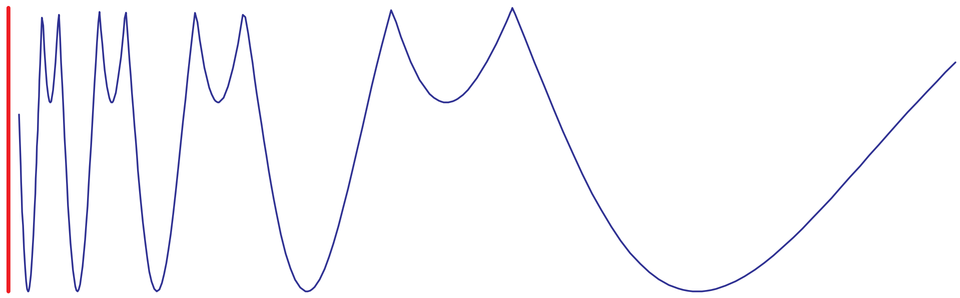


Figure 14.1.5: (A) M-Continuum

continua X that there are no nonempty proper subcontinua A and B of X with $X = A \cup B$. The first example of such a continuum was given by Brouwer in 1910 (see Figure B, where the first several steps of his construction, simplified by Z. Janiszewski (1888–1920) [2, p. 114] are presented). Finally in 1922 B. Knaster (1893–1980) gave a nice description of this continuum (with the first full proof of its indecomposability) in [3, p. 209-210]. Now the continuum is referred to as the simplest indecomposable continuum, or the horse-shoe continuum, or the B-J-K continuum (for Brouwer, Janiszewski and Knaster).

Finally the common boundary problem of plane domains has been solved in 1928 by K. Kuratowski (1896–1980) who proved in [4] and [5] that every plane continuum which is the common boundary of n open domains either is indecomposable or is the union of two indecomposable continua whenever $n \geq 3$, and when $n = 2$ it either is "monostratic" or has a natural "cyclic structure" in the sense that it is built up from layers naturally ordered in the same way as the individual points of the circle.

We can obtain a homeomorphic copy of the buckethandle continuum using the inverse limit

$$\varprojlim \{X_i, f_i\}_{i=1}^{\infty}$$

where for each $i = 1, 2, \cdots$ let $X_i = [0, 1]$ and $f_i(t) = 2t$ for $0 \leq t \leq 1/2$ and $f_i(t) = -2t + 2$ for $1/2 \leq t \leq 1$.

We can find more in [65, p.22] and [55, p.205].

[1] L. E. J. Brouwer : Zur Analysis Situs, Math. Ann. 68(1910), 422–434.

[2] Z. Janiszewski: Sur les continus irréductibles entre deux points, Journal de l'Ecole Polytechnique (2) 16(1912), 79–170.

[3] K. Kuratowski: Théorie des continus irréductibles entre deux points I, Fund. Math. 3(1922), 200–231.

[4] K. Kuratowski: Sur les coupures irréductibles du plan, Fund. Math. 6(1924), 130–145.

[5] K. Kuratowski: Sur la structure des frontières communes à deux regions, Fund. Math. 12(1928), 20–42.

[6] A. Schönflies: Beirträge zur Theorie der Punktmengen I, Math. Ann. 58(1904), 195–244; II 59(1904), 129–160; III 62(1906), 286–236.

14.1.7 Cantor Interaction

Cantor Interaction is a Charatonic's arcwise connected world ...

14.1.8 Pseudo-circle

Pseudo-circle is a strange continuum.

14.1.9 Semicircle Continuum

Semicircle Continuum is a nice continuum.

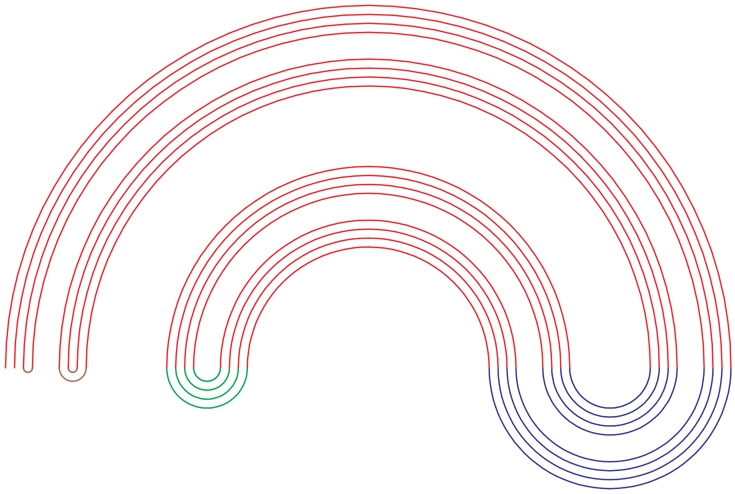


Figure 14.1.6: (A) Buckethandle

Let $A = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$;
for each $n = 1, 2, \cdots$ and each $k = 1, \cdots, 2^{n-1}$, let

$$B_{n,k} = \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{2k-1}{2^n} \right)^2 + y^2 = 4^{-n} \text{and} y \geq 0 \right\};$$

for each $n = 0, 1, \cdots$ and each $k = 1, \cdots, 3^n$, let

$$C_{n,k} = \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{2k-1}{2 \cdot 3^n} \right)^2 + y^2 = \frac{1}{4} \cdot 9^{-n} \text{and} y \leq 0 \right\};$$

let

$$X = A \cup \left[\bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{2^{n-1}} B_{n,k} \right) \right] \cup \left[\bigcup_{n=0}^{\infty} \left(\bigcup_{k=1}^{3^n} C_{n,k} \right) \right]$$

1) X is an example of an hlc continuum which is not regular.
2) X is the union of two regular continua.
See Nadler 10.59, p.193.

14.1.10 Boxes Continuum

Boxes Continuum is a nice continuum.
Let $A = \{(x, 2^{-n+1}) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$;
for each $n = 0, 1, 2, \cdots$ and each $m = 0, \cdots, 2^{n+1}$, let

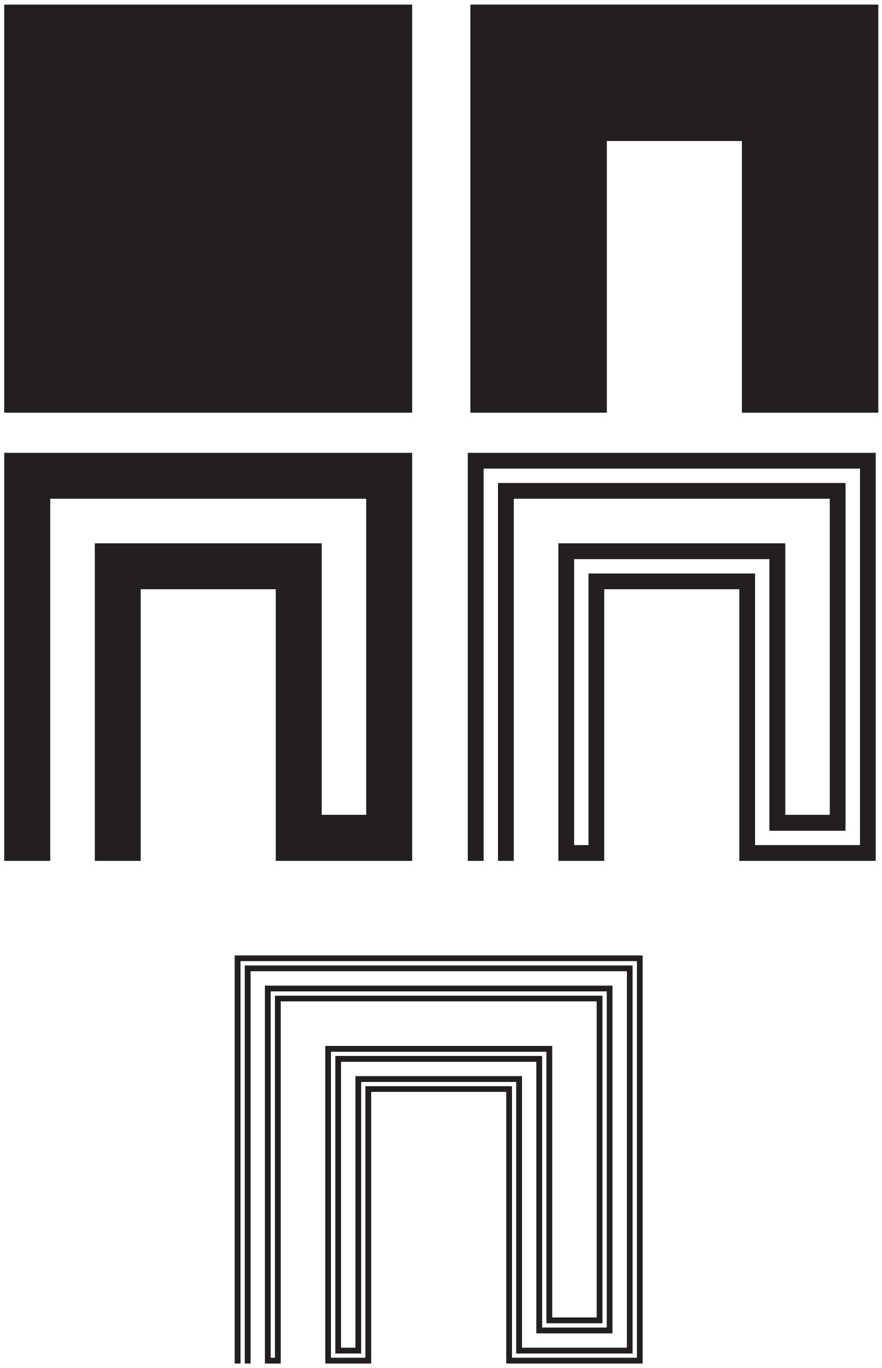


Figure 14.1.6: (B) several steps in the construction of the Buckethandle

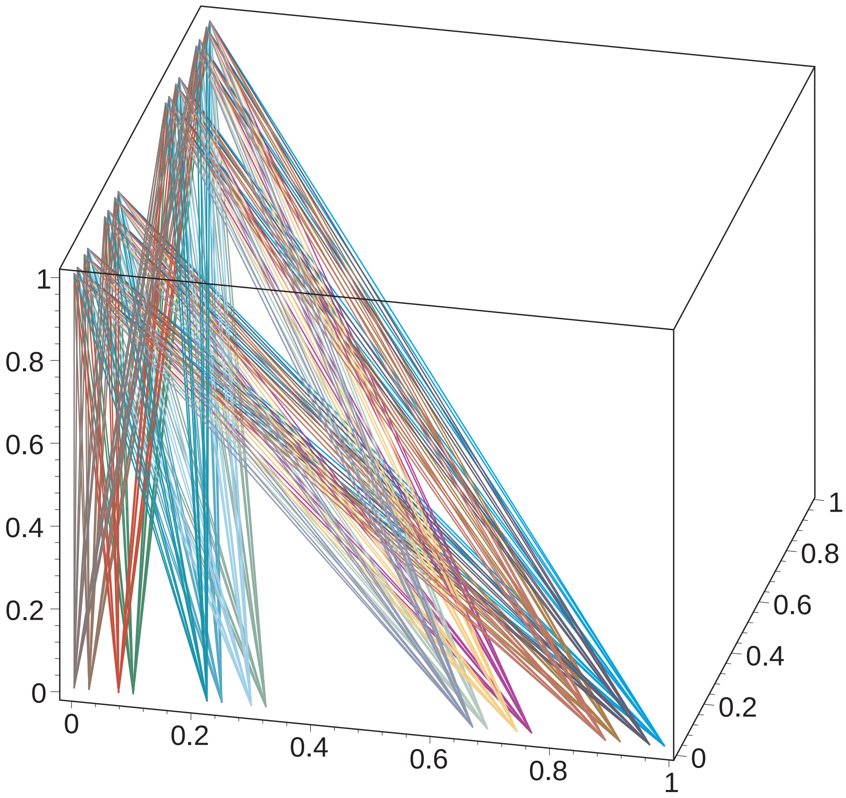


Figure 14.1.7: (A) Cantor Interaction

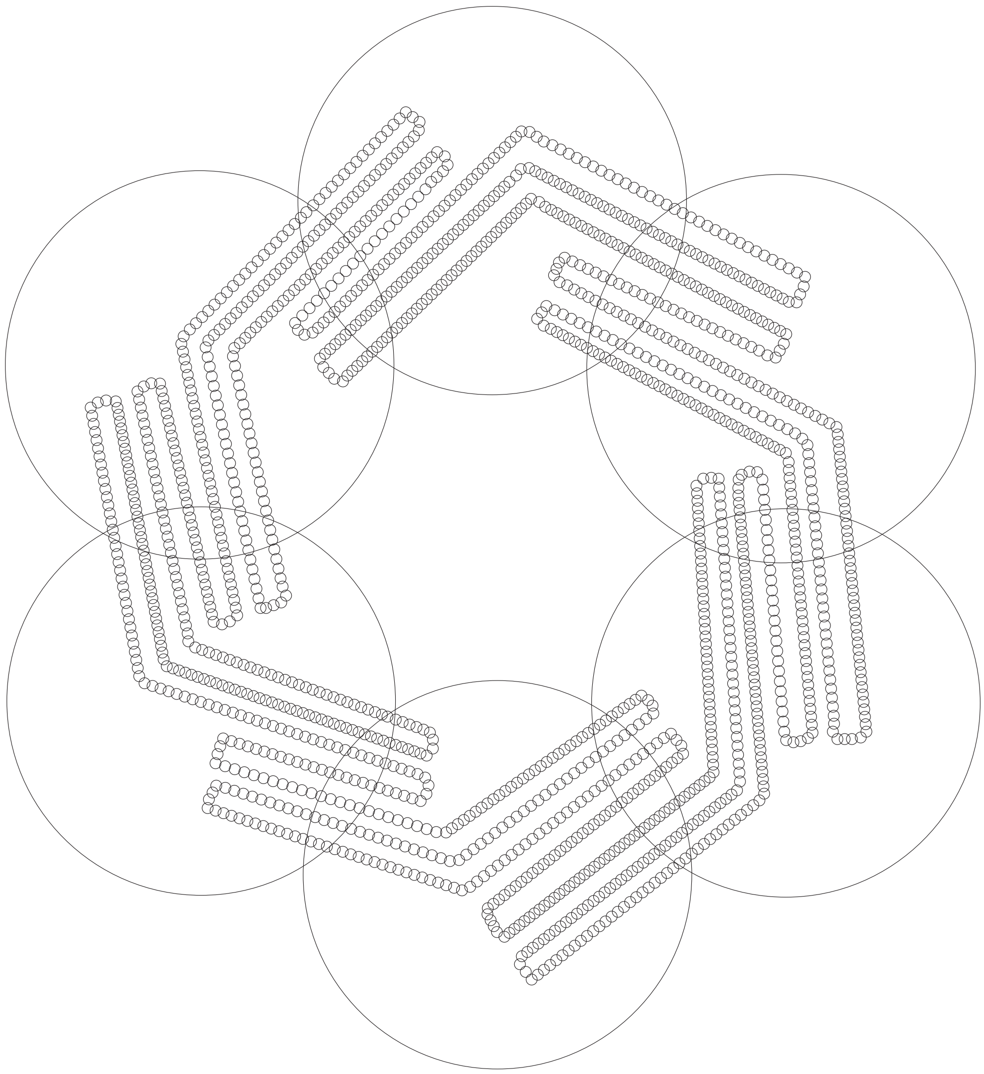


Figure 14.1.8: (A) A circular chain crooked in a circular chain with 6 links, construction of the Pseudo-circle

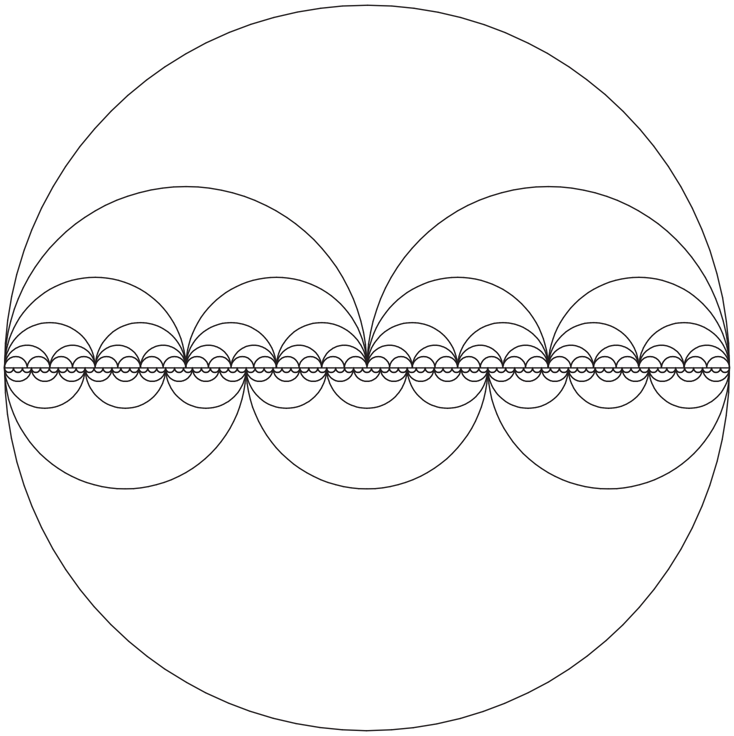


Figure 14.1.9: (A) Semicircle Continuum

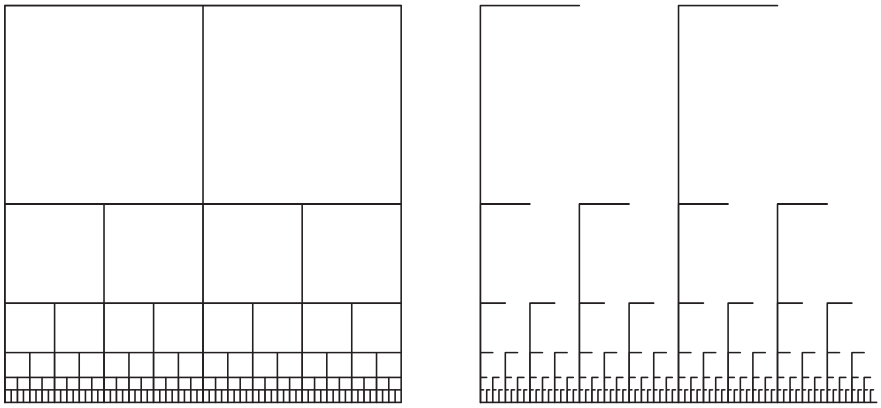


Figure 14.1.10: (A) Union of two dendrites, a non-hlc Boxes Continuum

$$B_{n,m} = \left\{(m \cdot 2^{-n-1}, y) \in \mathbb{R}^2 : 0 \leq y \leq 2^{-n}\right\};$$

finally, let

$$X = \left[\bigcup_{n=1}^{\infty} A_n\right] \cup \left[\bigcup_{n=0}^{\infty} \left(\bigcup_{m=0}^{2^{n+1}} B_{n,m}\right)\right].$$

1) X is an example of an non-hlc continuum which is not regular.
2) X is the union of two regular continua.
See Nadler 10.38, p.186.

14.2 New examples 2

New examples are nice.

14.2.1 Graph $K_{3,3}$

Graph $K_{3,3}$ is a not embedable in $\mathbb{R}^2$.

14.2.2 Graph K_5

Graph K_5 is a simple continuum.

14.2.3 Space Filling Curve

Space Filling Curve is a strange continuum.

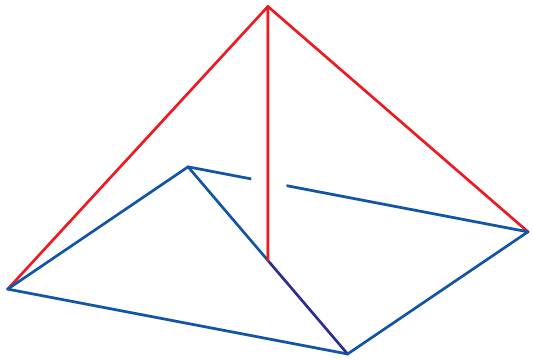


Figure 14.2.1: (A) Graph $K_{3,3}$

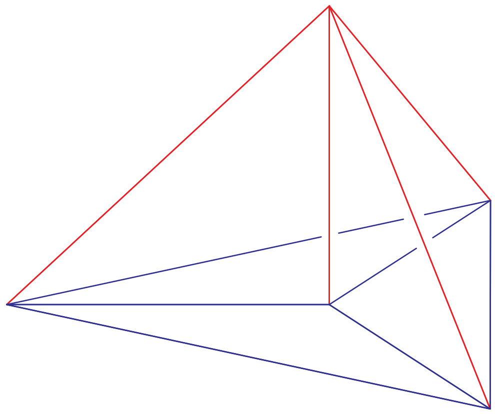


Figure 14.2.2: (A) Graph K_5

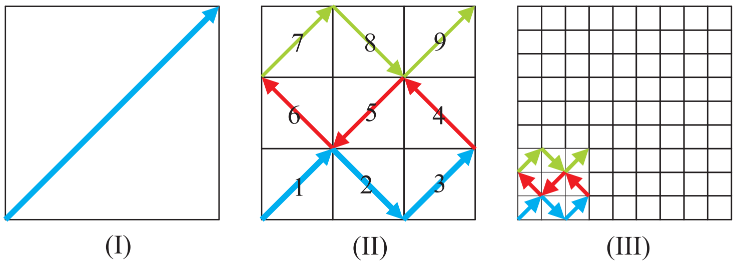


Figure 14.2.3: (A) An example of Space Filling Curve

14.2.4 Universal Arc-like Continuum

Universal Arc-like Continuum is a useful continuum.

14.2.5 Lakes of Wada

Lakes of Wada is a strange continuum.

14.2.6 Borsuk Fan

Borsuk Fan is a nice continuum.

14.2.7 Ingram's Atriodic Continuum

INGRAM COMMENT (modified):

Figure A represents the first and second composite of a mapping f of a triod $T = OA \cup OB \cup OC$, onto itself. The mapping f has the property that $f(O) = B$, $f(A) = C$, $f(B) = C$, $f(C) = C$, $f(C/2) = O$, $f(B/3) = O$, $f(B/2) = A/2$, $f(2B/3) = O$, $f(A/4) = O$, $f(A/2) = A$, $f(3A/4) = O$ and the mapping is piecewise linear. In the picture, the black square represents O, the green arm is the image of OA, the blue arm is the image of OB and the red arm is the image of OC. The inverse limit of the inverse sequence, T, f, is an atriodic tree-like continuum which is not chainable [Fund. Math., 77(1972), 99 - 107]. We call it an *Ingram's Atriodic Continuum.*

If one imagines that the fat triod in the top of the Fgure A represents the first of a sequence of tree chains covering the inverse limit then the blue-green-red folded triod represents the next terms of the sequence of covers.

14.2.8 Double Warsaw Circle

We call it a *Double Warsaw Circle.*

14.2.9 Warsaw Circle

We call it a *Warsaw Circle.*

14.2.10 Crooked Warsaw Circle

We call it a *Crooked Warsaw Circle.*

14.3 New examples 3

New examples 3 are useful.

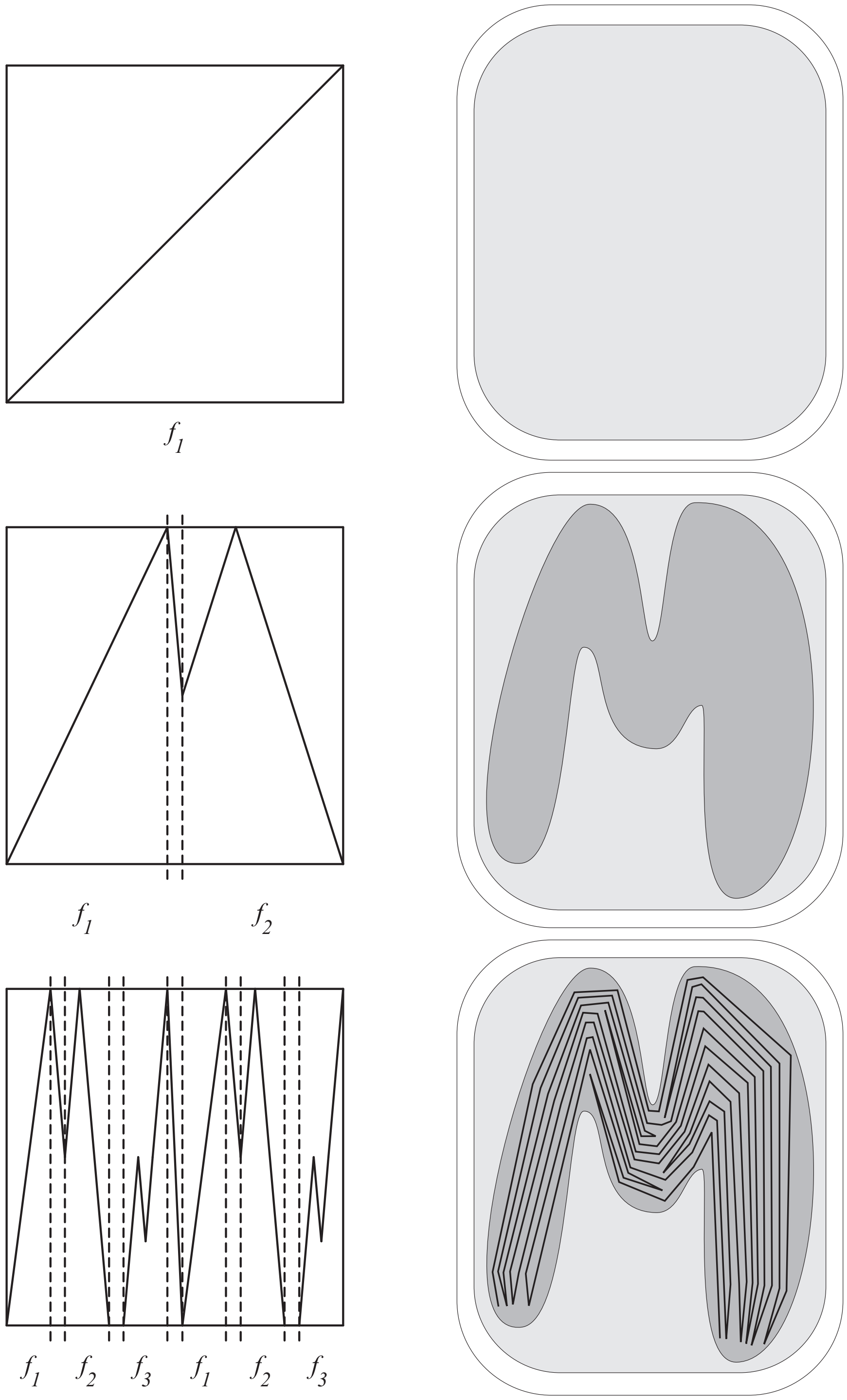
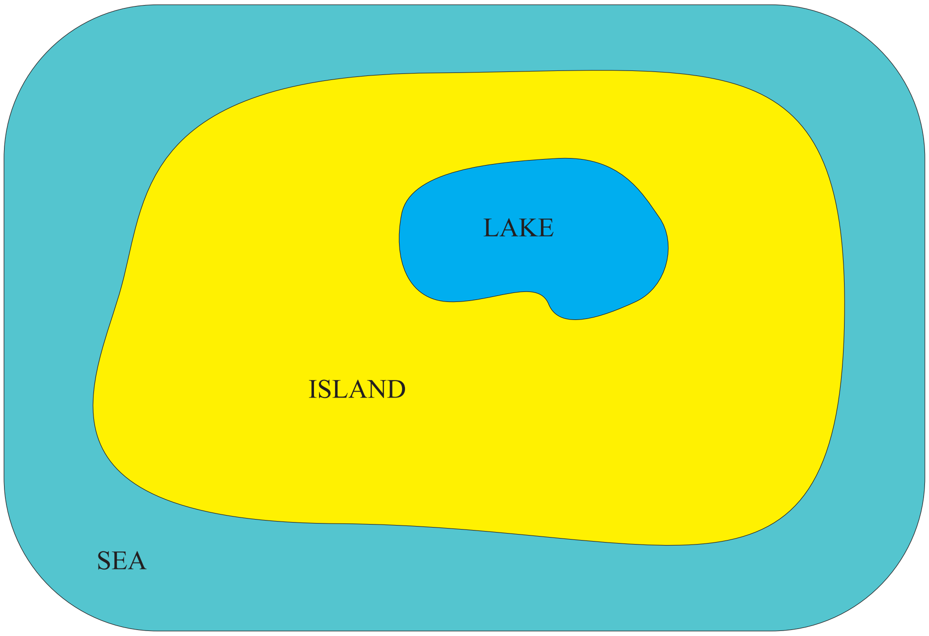
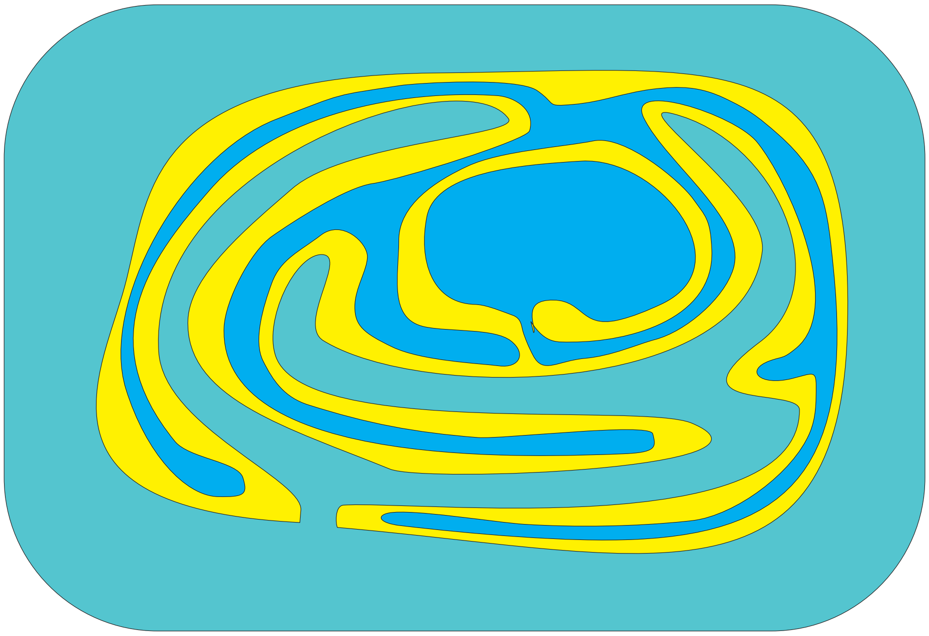


Figure 14.2.4: (A) First steps in the construction of the Universal Arc-like Continuum



(I)



(II)

Figure 14.2.5: (A) A construction of the Lakes of Wada

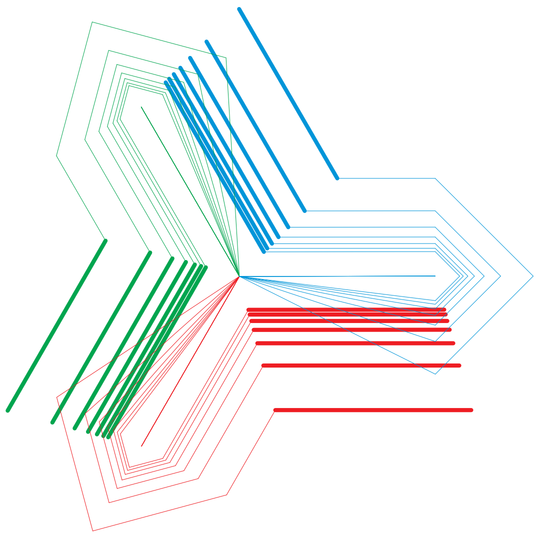


Figure 14.2.6: (A) thick segments are 'above' the thin ones, Borsuk Fan

14.3.1 Young Spiral

We call it a *Young Spiral*. See Roman Maňka : On spirals and fixed point property, Fund. Math. 144(1994), p. 1-9.

14.3.2 Young Space

We call it a *Young Space*. See Roman Maňka : On spirals and fixed point property, Fund. Math. 144(1994), p. 1-9.

14.3.3 Bing Space

We call it a *Bing Space*. See Roman Maňka : On spirals and fixed point property, Fund. Math. 144(1994), p. 1-9.

14.3.4 Universal Smooth Dendroid

We call it a *Universal Smooth Dendroid*.

14.3.5 Claytor's Curve A

We call it a *Claytor's Curve A*.

14.3.6 Claytor's Curve B

We call it a *Claytor's Curve B*.

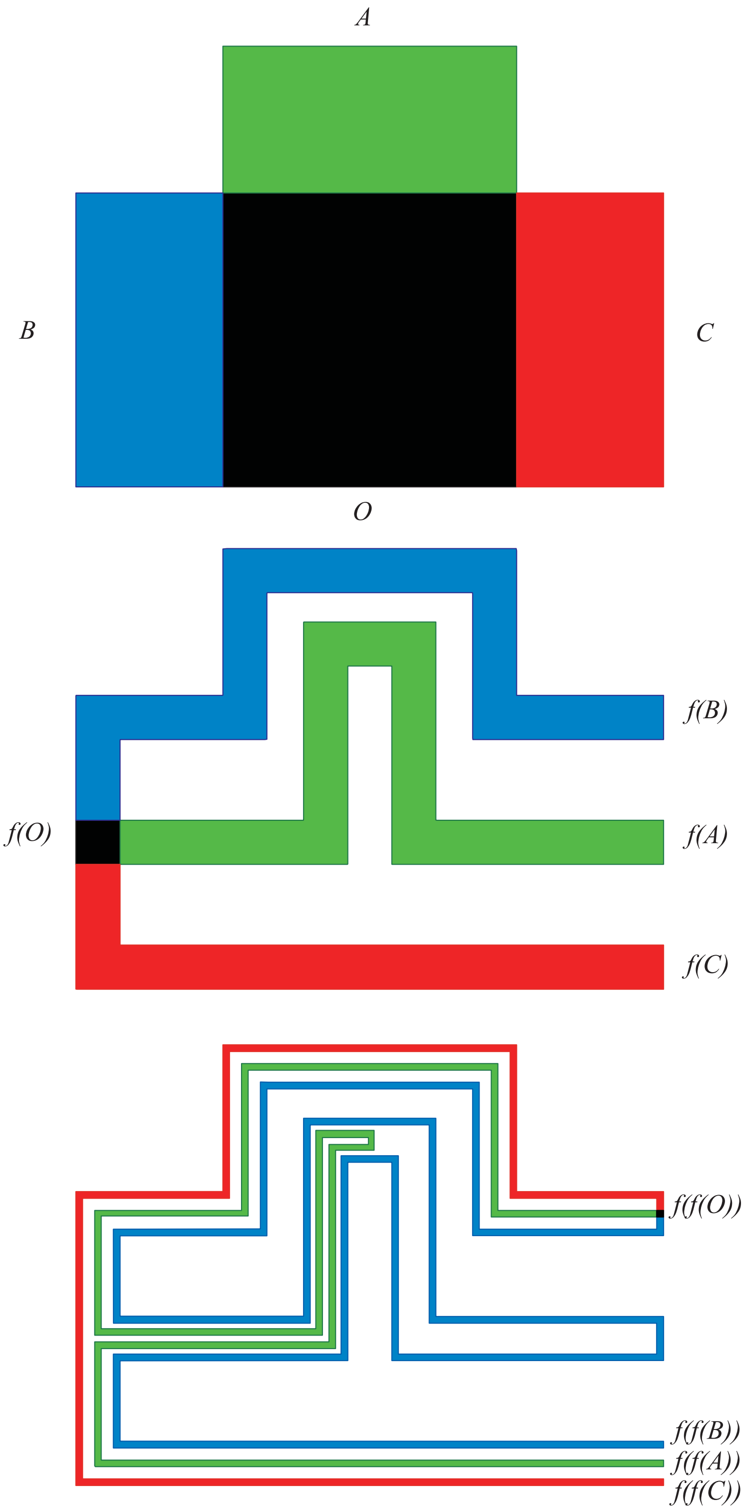


Figure 14.2.7: (A) First steps in the construction of the Ingram's Atriodic Continuum

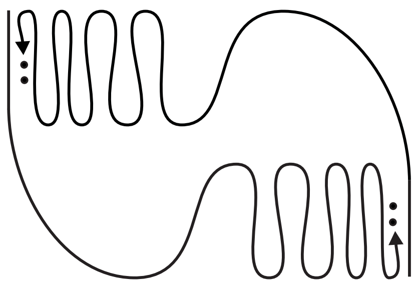


Figure 14.2.8: (A) Double Warsaw Circle

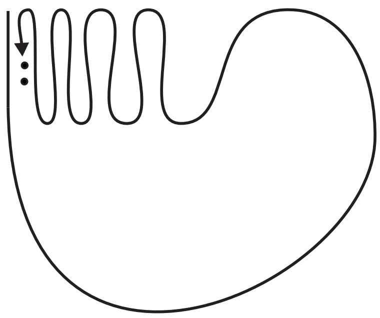


Figure 14.2.9: (A) Warsaw Circle

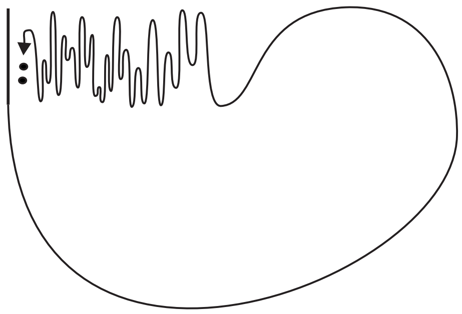


Figure 14.2.10: (A) Crooked Warsaw Circle

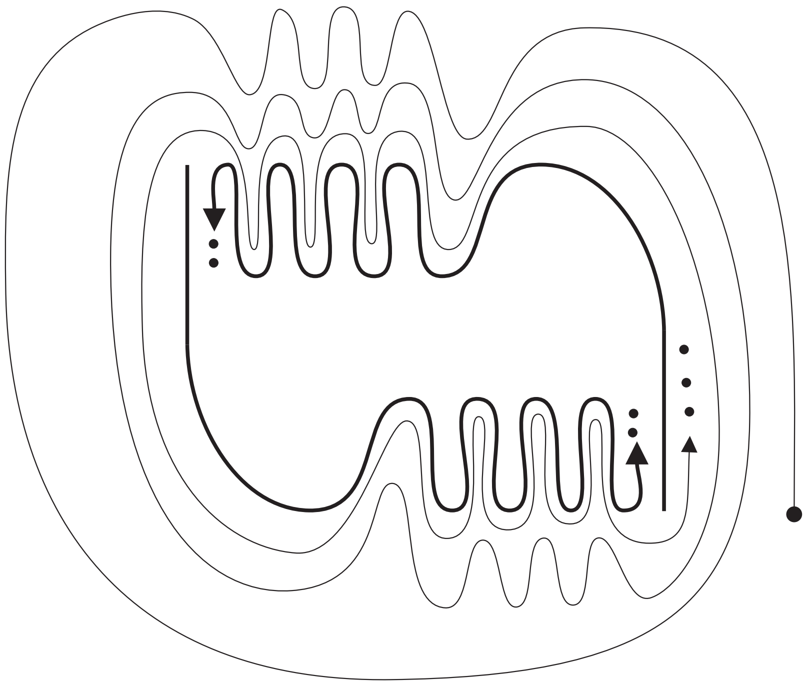


Figure 14.3.1: (A) Young Spiral

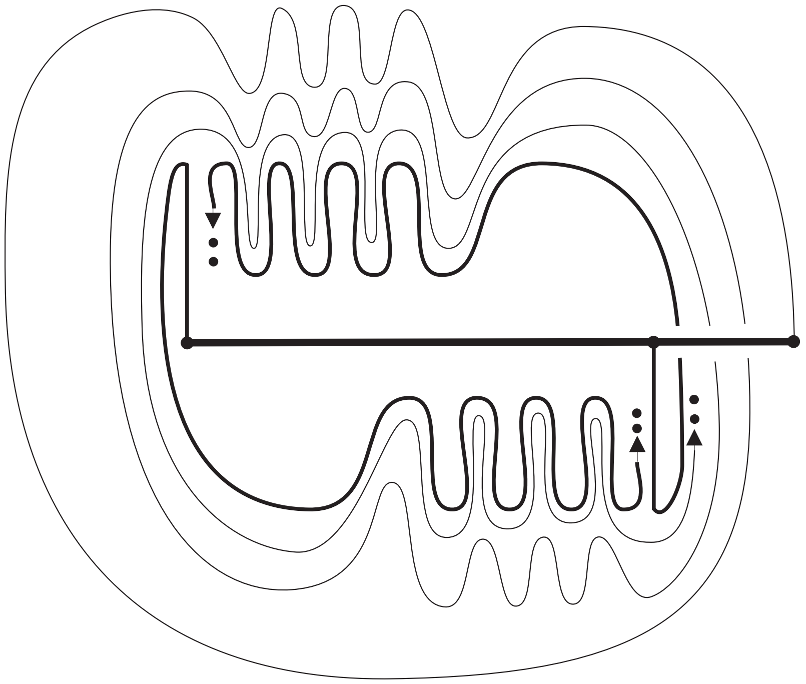


Figure 14.3.2: (A) Young Space

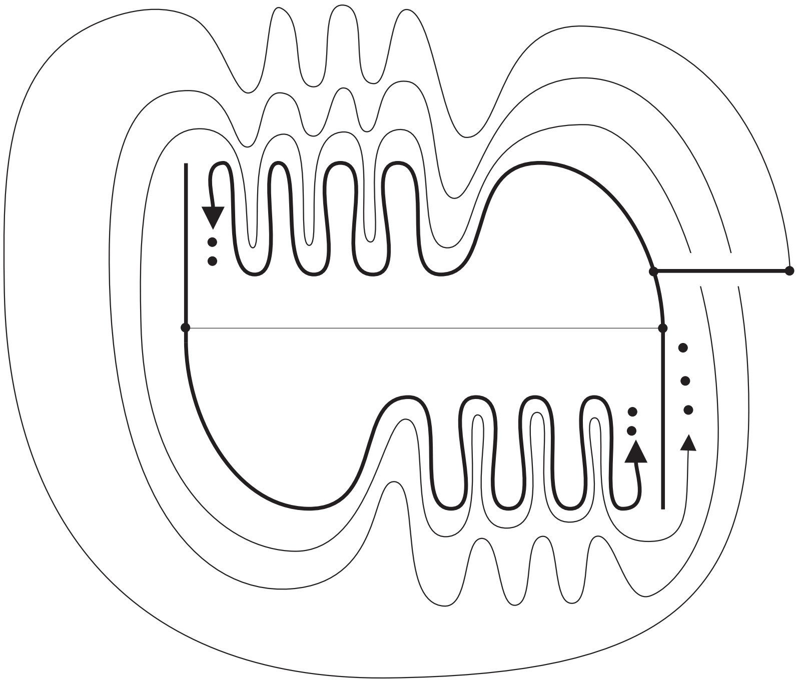


Figure 14.3.3: (A) Bing Space

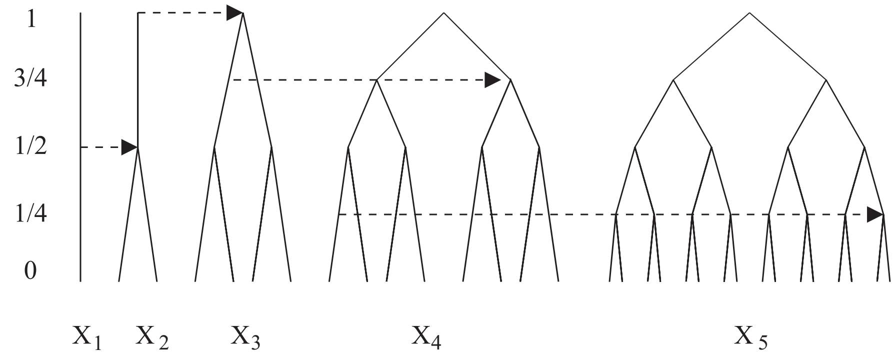


Figure 14.3.4: (A) Universal Smooth Dendroid

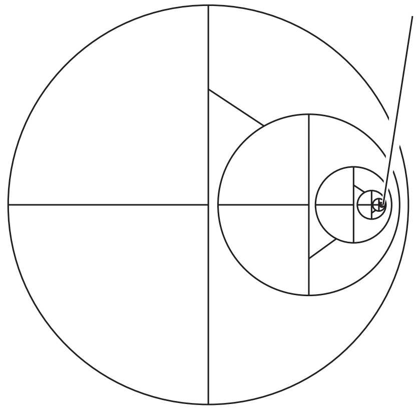


Figure 14.3.5: (A) Claytor's Curve A

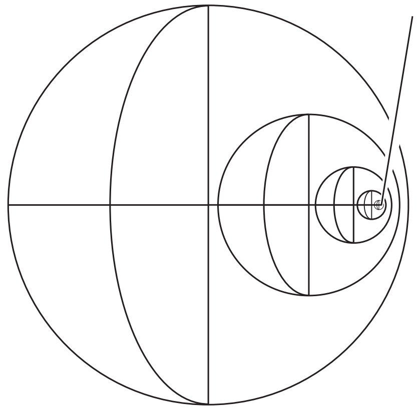


Figure 14.3.6: (A) Claytor's Curve B

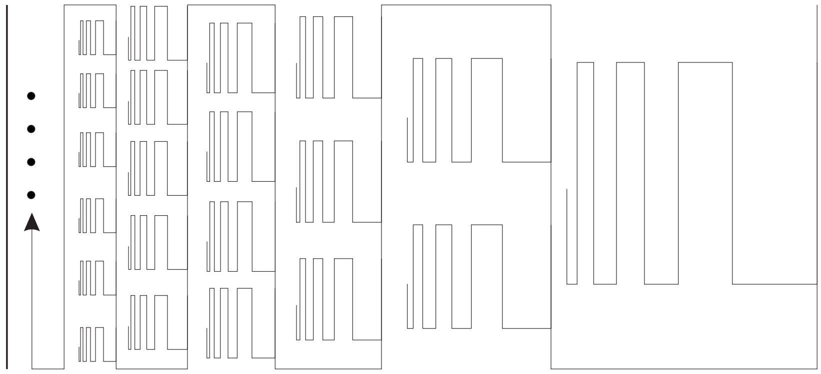


Figure 14.3.7: (A) Sin Addiction Continuum

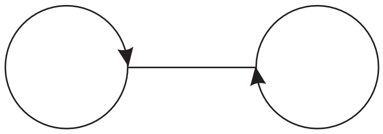


Figure 14.3.8: (A) Dumbbell

14.3.7 Sin Addiction Continuum

We call it a *Sin Addiction Continuum*.

14.3.8 Dumbbell

We call it a *Dumbbell*.

14.3.9 Figure Eight

We call it a *Figure Eight*.

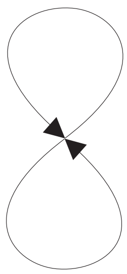


Figure 14.3.9: (A) Figure Eight

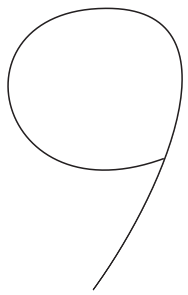


Figure 14.3.10: (A) Noose

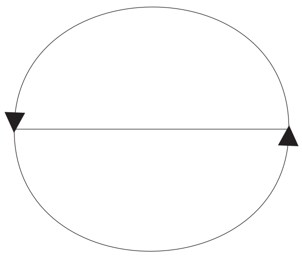


Figure 14.4.1: (A) Theta Curve

14.3.10 Noose

We call it a *Noose*.

14.4 New examples 4

New examples 4 are useful.

14.4.1 Theta Curve

We call it a *Theta Curve*.

14.4.2 Trapezoid Basins Continuum

We call it a *Trapezoid Basins Continuum*.

14.4.3 Dragon Continuum

We call it a *Dragon Continuum*. It is a fix point free continuum.

L. Fearnley, D. G. Wright : Geometric realization of a Bellamy Continuum, BUll. London Math. Soc. 15(1993), 177-183.

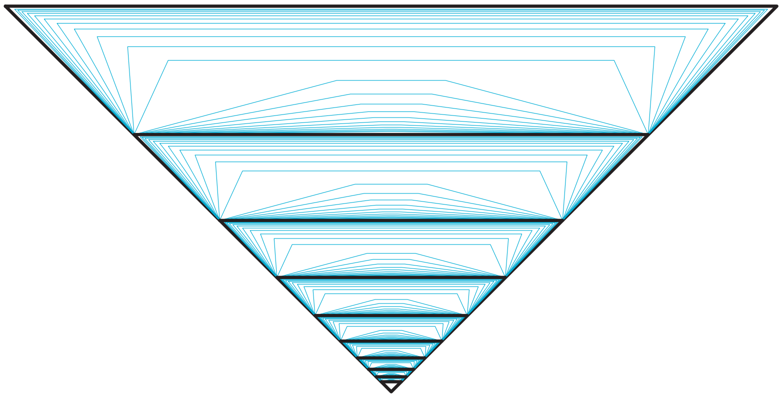


Figure 14.4.2: (A) Trapezoid Basins Continuum

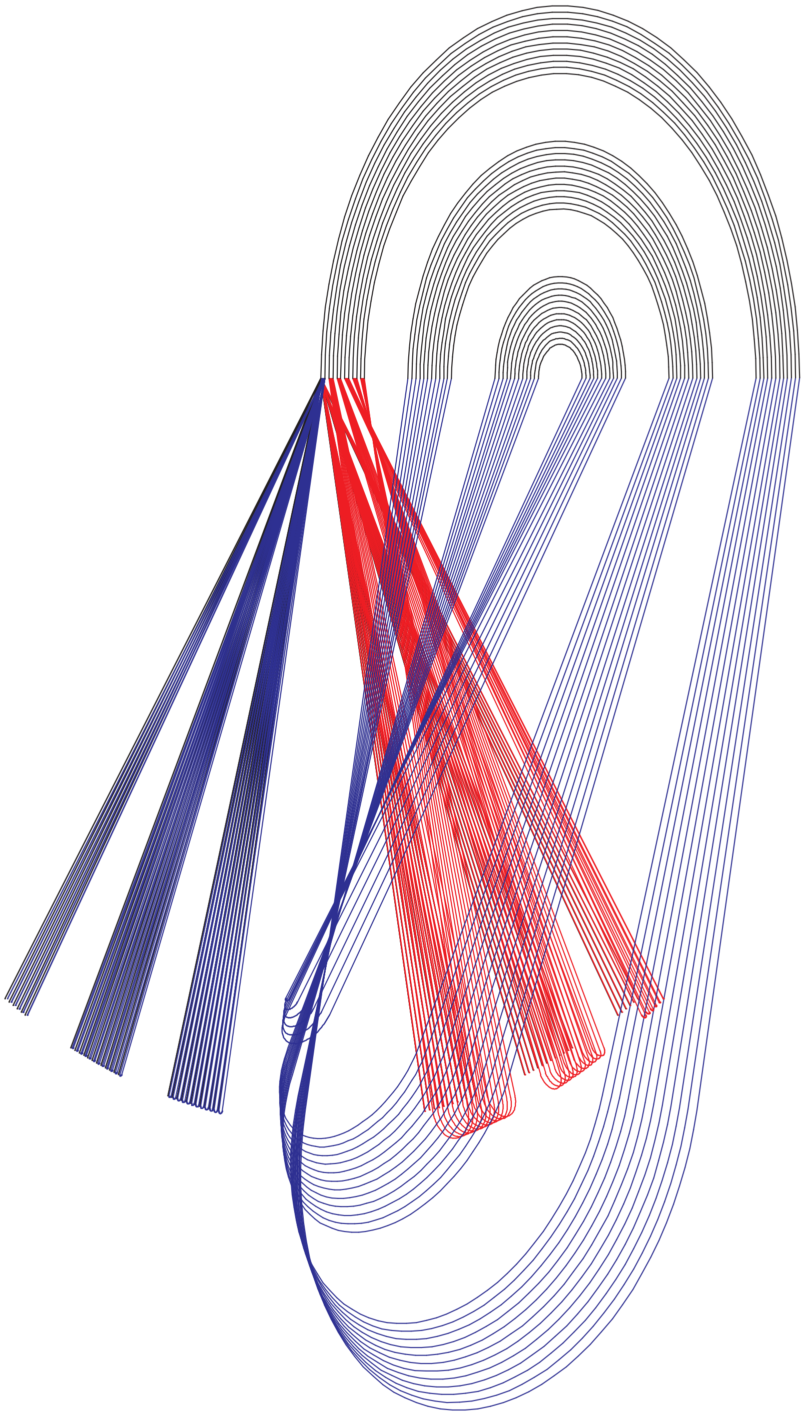
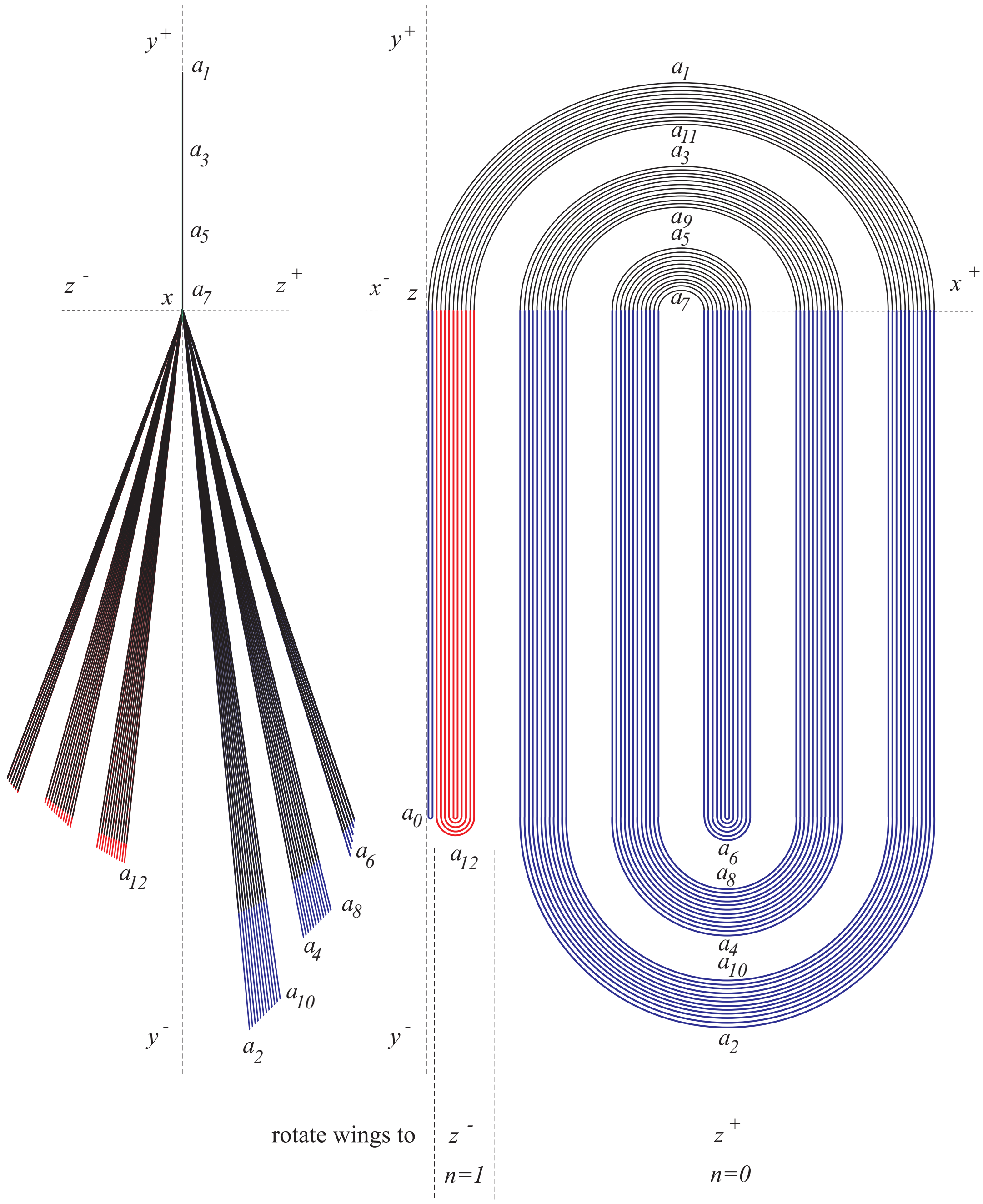


Figure 14.4.3: (A) Dragon Continuum



Fix point free mapping maps arc $a_n\ a_{n+1}$ to $a_{2n}\ a_{2(n+1)}$

Figure 14.4.3: (B) Dragon Continuum

Appendix A

Definitions

absolute end point

A point p of a continuum X is called an *absolute end point* of X if $X \setminus \{p\}$ is a composant of X.

absolute retract

An *absolute retract* is a space X such that if X is embedded as a closed subset X' of a space Y, then X' is a retract of Y.

absolutely terminal continuum

A proper subcontinuum K of a continuum X is said to be an *absolutely terminal continuum* of X provided that K is a terminal continuum of each subcontinuum L of X which properly contains K.

acyclic

A continuum X is said to be *acyclic* provided that each mapping from X into the unit circle $\mathbb{S}^1$ is homotopic to a constant mapping, i.e., for all mappings $f : X \to \mathbb{S}^1$ and $c : X \to \{p\} \subset \mathbb{S}^1$ there exists a mapping $h : X \times [0,1] \to \mathbb{S}^1$ such that for each point $x \in X$ we have $h(x,0) = f(x)$ and $h(x,1) = c(x)$.

almost chainable

A continuum M is *almost chainable* if, for each positive number ε, there exists an open cover D of M with mesh ε and a chain C of elements of D and an end link L of C such that no element of $D \setminus C$ intersects any link of C other than L and every point of M is at a distance less than ε from some link of C.

aposyndetic

A connected space X is *aposyndetic at* H with respect to K if there is a closed connected subset of X with H in its interior and not intersecting K, and X is *aposyndetic* if it is aposyndetic at each point with respect to every other point.

A continuum X is said to be *aposyndetic* provided that for each point $p \in X$ and for each $q \in X \setminus \{p\}$ there exists a subcontinuum K of X and an open set U of X such that $p \in U \subset K \subset X \setminus \{q\}$ (see e.g.[65, Exercise 1.22, p. 12]).

arc

An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$.

arc-smooth

Given a continuum X with an arc-structure A, the pair (X, A) (see arc-structure) is said to be *arc-smooth at a point* $v \in X$ provided that the induced function $A_v : X \to C(X)$ defined by $A_v(x) = A(v, x)$ is continuous. Then the point v is called an *initial point* of (X, A). The pair (X, A) is said to be *arc-smooth* provided that there exists a point in X at which (X, A) is arc-smooth. An arbitrary space X is said to be *arc-smooth at a point* $v \in X$ provided that there exists an arc-structure A on X for which (X, A) is arc-smooth at v. The space X is said to be *arc-smooth* if it is arc-smooth at some point (see [42, p. 546]). Note that a dendroid is smooth if and only if it is arc-smooth.

arc-structure

By an *arc-structure* on an arbitrary space X we understand a function $A : X \times X \to C(X)$ such that for every two distinct points x and y in X the set $A(x, y)$ is an arc from x to y and that the following metric-like axioms are satisfied for every points x, y and z in X:

(1) $A(x, x) = \{x\}$;

(2) $A(x, y) = A(y, x)$;

(3) $A(x, z) \subset A(x, y) \cup A(y, z)$,
with equality prevailing whenever $y \in A(x, z)$.

We put (X, A) to denote that the space X is equipped with an arc-structure A (see [42, p. 546]). Note that if there exists an arc-structure on a continuum, then the continuum is arcwise connected.

atomic

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]):
– *atomic* provided that for each subcontinuum K of X such that the set $f(K)$ is nondegenerate we have $K = f^{-1}(f(K))$;

atriodic

a continuum that contains no subcontinuum which is itself a triod is said to be *atriodic*

biconnected

A space is *biconnected* if it is not the sum of two mutually exclusive nondegenerate connected point sets.

bihomogeneous

Suppose X is a space such that, for each two points a and b, there is a homeomorphism of X onto itself such that $h(a) = b$ and $h(b) = a$. Then X is *bihomogeneous*.

Borel sets

The family F of *Borel sets* of a space S is the smallest family satisfying the conditions:
(i) Every closed set belongs to F;
(ii) If X is in F, then $S \setminus X$ is in F;
(iii) The countable intersection of elements of F belongs to F.

branch point

A *branch point* of a continuum is the vertex of a simple triod lying in that continuum.

chainable

A *simple chain* is a finite sequence $L_1, L_2, \cdots, L_n$ of open sets such that L_i intersects L_j if and only if $|i - j| \leq 1$. The terms of the sequence $L_1, L_2, \cdots, L_n$ are called the *links* of the chain. An ε-*chain* is a chain each of whose links has diameter less than ε. a continuum M is *chainable* if, for each positive number ε, M can be covered by an ε-chain.

chaotic

A nondegenerate topological space X is said to be *chaotic* if for any two distinct points p and q of X there exist an open neighborhood U of p and an open neighborhood V of q such that no open subset of U is homeomorphic to any open subset of V.

circle

A *circle* is any space which is homeomorphic to the unit circle.

completely regular

a continuum is *completely regular* if each nondegenerate subcontinuum has a nonempty interior.

composant

If a point p of a continuum X is given, then the *composant of* X *belonging to* p is defined to be the union of all proper subcontinua of X which contain p. A set is a *composant of* X provided it is the composant of X belonging to p for some point p of X.

confluent

A mapping f of a compact space X onto a compact space Y is *confluent* if, for each continuum C in Y, each component of $f^{-1}(C)$ is mapped onto C by f.

-connected

A topological space (X, τ) is said to be σ-*connected* provided that X is not the union of more than one and at most countably many nonempty, mutually disjoint, closed subsets.

A continuum is δ-*connected* if each two of its points can be connected by an hereditarily decomposable irreducible subcontinuum.

connected im kleinen

A space is *connected im kleinen* if it is aposyndetic at each point with respect to each closed point set not containing that point.

continuous selection of

Let a continuum X, a compact space Y and a function $F : X \to 2^Y$ be given. A function $f : X \to Y$ is called a *continuous selection of* F provided that it is continuous and $f(x) \in F(x)$ for each $x \in X$.

converges homeomorphically to the continuum

The statement that the sequence $M_1, M_2, \cdots$ *converges homeomorphically to the continuum* M means there exists a sequence $h_1, h_2, \cdots$ of homeomorphisms such that, for each positive integer i, h_i is a homeomorphism from M_i onto M and for each positive number ε there exists a positive integer N such that if $j > N$ then, for all x, $\mathrm{dist}(h_j(x), x) < \varepsilon$.

converge 0-regularly

A sequence of sets $\{A_n\}$ contained in a metric space X with a metric d is said to *converge* 0*-regularly* to its limit $A = \mathrm{Lim}\, A_n$ (see hyperspace) provided that for each $\varepsilon > 0$ there is a $\delta > 0$ and there is an index $n_0 \in \mathbb{N}$ such that if $n > n_0$ then for every two points $p, q \in A_n$ with $d(p, q) < \delta$ there is a connected set $C_n \subset A_n$ satisfying conditions $p, q \in C_n$ and $diam_n < \varepsilon$ (see [78, Chapter 9, §3, p. 174]).

convex

Given a continuum X with an arc-structure A, a subset Z of X is said to be *convex* provided that for each pair of points x and y of Z the arc $A(x, y)$ is a subset of Z. If Z is a convex subcontinuum of X, then $A|Z \times Z$ is an arc-structure on Z. We define X to be *locally convex at a point* $p \in X$ provided that for each open set U containing p there is a convex set Z such that $p \in \mathrm{int}\, Z \subset \mathrm{cl}\, Z \subset U$ (see [42, I.2, p. 548-549]).

continuum

A *continuum* is a nonempty, compact, connected metric space.

contractible

A continuum X is said to be *contractible* provided there is a homotopy $h : X \times [0,1] \to X$ such that for some point $p \in X$ we have $h(x,0) = x$ and $h(x,1) = p$ for each $x \in X$ (see e.g. [64, (16.2), p. 532]).

curve

A *curve* is any one-dimensional continuum.

cut point

Let $S, T)$ be a connected topological space, and let $p \in S$. If $S \setminus \{p\}$ is connected, then p is called a *non-cut point of* S. If $S \setminus \{p\}$ is not connected, then p is called a *cut point of* S.

C^*-smooth

Let X be a continuum. Define $C^* : C(X) \to C(C(X))$ by $C^*(A) = C(A)$. It is known that for any continuum X the function C^* is upper semi-continuous, [64, Theorem 15.2, p. 514], and it is continuous on a dense G_δ subset of $C(X)$, [64, Corollary 15.3, p. 515]. A continuum X is said to be *C^*-smooth at* $A \in C(X)$ provided that the function C^* is continuous at A. A continuum X is said to be *C^*-smooth* provided that the function C^* is continuous on $C(X)$, i.e., at each $A \in C(X)$ (see [64, Definition 5.15, p. 517]).

cyclic element

Let a continuum X be locally connected and let $p \in X$. By a *cyclic element* of X containing p we mean either the singleton $\{p\}$ if p is a cut point or an end point of X, or the set consisting of p and of all points $x \in X$ such that no point of X cuts X between p and x, otherwise (see e.g. [78, Chapter 4, §2, p. 66]).

decomposable

a continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua.

deformation retract of

Let X be a space and let A and B be subspaces of X with $A \subset B$. Then A is called a *deformation retract of* B over X provided that the identity mapping

$i_B : B \to B$ is homotopic in X to a retraction $r : B \to A$. Further, A is called a *strong deformation retract of* B over X provided that it is a deformation retract of B over X and the homotopy keeps the points of A fixed throughout the entire deformation of B into A (see e.g. [41, Definition 6.3, p. 324]).

dendroid

A *dendroid* is an arcwise connected, hereditarily unicoherent continuum.

dendrite

A *dendrite* means a locally connected continuum containing no simple closed curve.

dispersion point

A point p of a nondegenerate connected space X is an *explosion point*, or *dispersion point*, of X provided that $X \setminus \{p\}$ is totally disconnected.

end point

Point of order 1 in a continuum X is called *end point* of X; the set of all end points of X is denoted by $E(X)$.

equivalent

Let $\mathfrak{M}$ be a class of mappings. Two spaces X and Y are said to be *equivalent with respect to* $\mathfrak{M}$ (shortly $\mathfrak{M}$-*equivalent*) if there are two mappings, both in $\mathfrak{M}$, one from X onto Y and the other from Y onto X. If $\mathfrak{M}$ means the class of monotone mappings, we say that X and Y are *monotonely equivalent*.

F-equivalent

If X and Y are continua, we say that Y is F-*equivalent* to X provided there is a mapping in F from X onto Y and a mapping in F from Y onto X.

feebly monotone

Let X and Y be continua. A mapping $f : X \to Y$ is said to be *feebly monotone* provided that if A and B are proper subcontinua of Y such that $Y = A \cup B$, then their inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected.

finitely Suslinean

a continuum X is *finitely Suslinean* if, for each positive number ε, X does not contain infinitely many mutually exclusive subcontinua of diameter greater than ε.

finitely linear

A function $f I \to Y$, where I is a closed interval of the real line, is called *finitely linear* provided there exists a positive integer m such that I can be decomposed, for each $\varepsilon > 0$, into a finite number of closed subintervals $I_1, I_2, \cdots, I_k$ each of length less than ε and with the property that the set $f(I_i)$ meets at most m of the sets $f(I_1), f(I_2), \cdots, f(I_k)$ for $i = 1, 2, \cdots, k$.

fixed point

We say that a function $F : X \to 2^X$ (or a function $F : X \to C(X)$) *has a fixed point* provided that there is a point $x \in X$ such that $x \in F(x)$.

fixed set property

A topological space X is said to have the *fixed set property* for a certain class C of maps of X onto itself provided there exists, for each non-empty closed set A in X, a map f in C such that $f(x) = x$ if and only if x is in A.

fixed point self-homeomorphic

A topological space X is called *fixed point self-homeomorphic* if for any point p, any neighborhood U of p, there is an embedding $h : X \to U$ with $h(p) = p$ and $p \in \operatorname{int} h(X)$.

fixed ball self-homeomorphic

A topological space X is called *fixed ball self-homeomorphic* if for any point p, any neighborhood U of p, there is a neighborhood V of p with $V \subseteq U$ and an embedding $h : X \to U$ satisfying $h|_V = \mathrm{id}_V$.

free arc

A *free arc* in X is any subset of X of the form $A \setminus \{x, y\}$ where A is an arc in X with end points x and y and $A \setminus \{x, y\}$ is open in X.

graph

A *graph* is a continuum which can be written as the union of finely many arcs any two of which are either disjoint or intersect only in one or both their end points.

half-ray curve

a continuum is a *half-ray curve* if it is a continuous 1-1 image of the nonnegative reals.

hereditary

A property of a continuum X is said to be *hereditary* if every subcontinuum of X has the property. In particular, a continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected.

hereditarily

Let $\mathfrak{M}$ be a class of mappings between continua. A mapping $f : X \to Y$ between continua is said to be *hereditarily* $\mathfrak{M}$ provided that its restriction to any subcontinuum of the domain X is in $\mathfrak{M}$ (see [59, Chapter 4, Section B, p. 16]).

hereditarily decomposable

a continuum X is said to be *hereditarily decomposable* provided that each non-degenerate subcontinuum of X (as well as X itself) is decomposable.

hereditarily equivalent

a continuum is *hereditarily equivalent* if it is homeomorphic to each of its non-degenerate subcontinua.

hereditarily indecomposable

a continuum X is said to be *hereditarily indecomposable* provided that each of its subcontinua is indecomposable, that is, for each subcontinuum $C \subset X$ and for every continua A and B such that $A \cup B = C$ we have either $A = C$ or $B = C$.

hereditarily locally connected

A continuum X is said to be *hereditarily locally connected*, written *hlc*, provided that every subcontinuum of X is a Peano Continuum.

hereditarily unicoherent

A connected topological space is said to be *hereditarily unicoherent* provided that each of its closed, connected subsets is unicoherent.

homogeneous

Let $\mathfrak{M}$ be a class of mappings. A space X is said to be *homogeneous with respect to* $\mathfrak{M}$ (or shortly $\mathfrak{M}$*-homogeneous*) provided that for every two points p and q of X there is a surjective mapping $f : X \to X$ such that $f(p) = q$ and $f \in \mathfrak{M}$. If $\mathfrak{M}$ is the class of homeomorphisms, we get the concept of a *homogeneous* space.

HU-terminal

A subcontinuum K of a hereditarily unicoherent continuum X is said to be a *HU-terminal continuum* of X provided that K is contained in an irreducible subcontinuum of X and for every irreducible subcontinuum I of X containing K there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$.

hyperspace

Given a continuum X with a metric d, we let 2^X to denote the *hyperspace* of all nonempty closed subsets of X equipped with the *Hausdorff metric* H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\} \quad \text{for } A, B \in 2^X$$

(see e.g. [64, (0.1), p. 1 and (0.12), p. 10]). If $H(A, A_n)$ tends to zero as n tends to infinity, we put $A = \operatorname{Lim} A_n$. Further, we denote by $F_1(X)$ the hyperspace of singletons of X, and by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. Since X is homeomorphic to $F_1(X)$, there is a natural embedding of X into $C(X)$, and so we can write $X \subset C(X) \subset 2^X$. Thus one can consider a retraction from either $C(X)$ or 2^X onto X.

indecomposable

a continuum X is said to be *indecomposable* provided that X cannot be written as the union of two proper subcontinua.

induced mapping

Let $f : X \to Y$ be a mapping between continua. Then the *induced mapping* $C(f) : C(X) \to C(Y)$ is defined by $C(f)(A) = f(A)$, where A in the left member of the equality means an element of $C(X)$, while in the right one it is understood as a subcontinuum of X (see [64, (0.49), p. 23]).

inverse limit

The reader can find necessary information on the inverse limits of inverse sequences e.g. in Section 2 of the second chapter of [65], as well as in [55].

irreducible

Let X be a continuum and $A \subset X$. Then, X is said to be *irreducible about* A provided no proper subcontinuum of X contains A. a continuum X is said to be *irreducible* provided that X is irreducible about $\{p, q\}$ for some $p, q \in X$.

light

A mapping is *light* if each point inverse is totally disconnected.

like

If X is a metric space, a mapping f from X to a space Y is an ε*-map* if, for each point y of Y, $diam f^{-1}(y)) \leq \varepsilon$. If C is a collection of continua, a continuum M is C*-like* if, for every positive number ε, there exists an ε-map of M onto an element of C. In particular, a continuum is *tree-like* if, for some collection C of trees, M is C-like.

A concept of a tree-like continuum can be defined in several (equivalent) ways. One of them is the following. A continuum X is said to be *tree-like* provided that for each $\varepsilon > 0$ there is a tree T and a surjective mapping $f : X \to T$ such that f is an ε-mapping (i.e., $diam^{-1}(y) < \varepsilon$ for each $y \in T$). Let us mention that a continuum X is tree-like if and only if it is the inverse limit of an inverse sequence of trees with surjective bonding mappings. Compare e.g. [65, p. 24].

Using a concept of a nerve of a covering, one can reformulate the above definition saying that a continuum X is be tree-like provided that for each $\varepsilon > 0$ there is an ε-covering of X whose nerve is a tree.

Finally, the original definition using tree-chains can be found e.g. in Bing's paper [4, p. 653].

locally

Let $\mathfrak{M}$ be a class of mappings between compact spaces. A surjective mapping $f : X \to Y$ between continua is said to be *locally* $\mathfrak{M}$ provided that for each point $x \in X$ there is a closed neighborhood V of x such that $f(V)$ is is a closed neighborhood of $f(x)$ and that the restriction $f|V$ is in $\mathfrak{M}$ (see [59, Chapter 4, Section C, p. 18]).

locally confluent

A mapping f from a space X onto a space Y is *locally confluent* if, for each point y of Y, there is an open set U containing y such that $f^{-1}(\overline{U})$ is confluent.

local homeomorphism

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]):
– a *local homeomorphism* provided that for each point $x \in X$ there exists a an open neighborhood U of x such that $f(U)$ is an open neighborhood of $f(x)$ and that f restricted to U is a homeomorphism between U and $f(U)$.

local separating point

A point p of a locally compact separable metric space L is a *local separating point* of L provided there exists an open set U of L containing p and two points x and y of the component containing p of U such that $U \setminus \{p\}$ is the sum of two mutually separated point sets, one containing x and the other containing y.

MO-mapping

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]):
– an *MO-mapping* provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$ such that f_1 is open and f_2 is monotone.

monotone

A continuous function $f : X \to Y$ is said to be *monotone* provided $f^{-1}(y)$ is connected for each $y \in Y$.

mutually aposyndetic

a continuum M is *mutually aposyndetic* if, for each two points A and B of M, there exist mutually exclusive subcontinua H and K of M containing A and B, respectively, in their interiors. a continuum is *strictly non-mutually aposyndetic* if each two of its subcontinua with interiors intersect.

natural projection

Let continua X and Y be given. A mapping $f : X \times Y \to X$ is called the *natural projection* provided that it is defined by $f((x, y)) = x$.

neat

A dendroid X is said to be *neat* provided that each one of its subdendroids has no improper shore point (see [71, p. 939]).

Let $\mathfrak{M}$ be a class of mappings. A class $\mathfrak{M}$ of mappings is said to be *neat* provided that if all homeomorphisms are in $\mathfrak{M}$ and the composition of any two mappings in $\mathfrak{M}$ is also in $\mathfrak{M}$.

near homeomorphism

A mapping $f : X \to Y$ is called a *near homeomorphism* provided that for each $\varepsilon > 0$ there is a homeomorphism $h : X \to Y$ such that $\sup\{d(f(x), h(x)) : x \in X\} < \varepsilon$.

OM-mapping

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]):
– an *OM-mapping* provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$ such that f_1 is monotone and f_2 is open.

open

A mapping f of a space X into a space Y is *open* if, for each open A in X the image $f(A)$ is open in Y.

openly minimal

We say that a dendrite X is *openly minimal* provided that every open image of X can be openly mapped onto X.

orbit

Given a space X let $\mathcal{H}(X)$ stand for the group of autohomeomorphisms of X. If a point $p \in X$ is fixed, then $\{h(p) \in X : h \in \mathcal{H}(X)\}$ is called an *orbit of* p. Orbits of points of X either are mutually disjoint or coincide, and their union is the whole X.

order

Let (X,T) be a topological space, and let $A \subset X$. Let β be a cardinal number. We say that A *is of order less than equal to* β in X, written $\mathrm{ord}(A,X) \leq \beta$, provided that for each $U \in T$ such that $A \subset U$, there exists $V \in T$ such that $A \subset V \subset U$ and $|Bd(V)| \leq \beta$.

We say that A *is of order* β *in* X, written $\mathrm{ord}(A,X) = \beta$, provided that $\mathrm{ord}(A,X) \leq \beta$ and $\mathrm{ord}(A,X) \not\leq \alpha$ for any cardinal number $\alpha < \beta$.

A concept of an *order* of a point p in a continuum X (in the sense of Menger-Urysohn), written $\mathrm{ord}\,(p,X)$, is defined as follows. Let $\mathfrak{n}$ stand for a cardinal number. We write:

$\mathrm{ord}\,(p,X) \leq \mathfrak{n}$ provided that for every $\varepsilon > 0$ there is an open neighborhood U of p such that *diam* $\leq \varepsilon$ and $\mathrm{card}\,\mathrm{bd}\,U \leq \mathfrak{n}$;

$\mathrm{ord}\,(p,X) = \mathfrak{n}$ provided that $\mathrm{ord}\,(p,X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\mathrm{ord}\,(p,X) \leq \mathfrak{m}$ does not hold;

$\mathrm{ord}\,(p,X) = \omega$ provided that the point p has arbitrarily small open neighborhoods U with finite boundaries $\mathrm{bd}\,U$ and $\mathrm{card}\,\mathrm{bd}\,U$ is not bounded by any $n \in \mathbb{N}$.

Thus, for any continuum X we have

$$\mathrm{ord}\,(p,X) \in \{1, 2, \ldots, n, \ldots, \omega, \aleph_0, 2^{\aleph_0}\}$$

(convention: $\omega < \aleph_0$); see [55, §51, I, p. 274].

Let a dendroid X and a point $p \in X$ be given. Then p is said to be a *point of order at least* $\mathfrak{m}$ *in the classical sense* provided that p is the center of an $\mathfrak{m}$-od contained in X. We say that p is a *point of order* $\mathfrak{m}$ *in the classical sense* provided that $\mathfrak{m}$ is the minimum cardinality for which the above condition is satisfied (see [8, p. 229]).

order preserving mapping

For each point p of a continuum X equipped with an arc-structure $A : X \times X \to C(X)$ we define a partial order $\leq_p$ by letting $x \leq_p y$ whenever $A(p,x) \subset A(p,y)$. Let X and Y be continua with fixed arc-structures A and B, respectively. We say that a surjective mapping $f : X \to Y$ is a $\leq_p$*-mapping* provided that $x \leq_p y$ in X implies that $f(x) \leq_{f(p)} f(y)$ in Y. If, in addition, $Y \subset X$, $B = A|(Y \times Y)$, and f is a retraction, then f is called a $\leq_p$*-retraction* (or $\leq_p$*-preserving retraction*). The concept of a $<_p$*-mapping* is defined in a similar manner (with $f(x) \neq f(y)$ implied by $x \neq y$). For *order preserving mappings* see e.g. [42, I.7, p. 553].

ordinary point

Point of order 2 is called *ordinary point* of X; the set of all ordinary points of X is denoted by $O(X)$.

periodic point of

Let a continuum X and a mapping $f : X \to X$ be given. For each natural number n denote by f^n the n-th iteration of f. A point $p \in X$ is called a *periodic point of* f provided that there is $n \in \mathbb{N}$ such that $f^n(p) = p$ The set of periodic points of a mapping $f : X \to X$ are denoted by $P(f)$.

periodic-recurrent property

The set of periodic points and of recurrent points of a mapping $f : X \to X$ are denoted by $P(f)$ and $R(f)$ respectively. Clearly $P(f) \subset R(f)$. A continuum X is said to have the *periodic-recurrent property* (shortly *PR-property*) provided that for every mapping $f : X \to X$ the equality $\mathrm{cl}\, P(f) = \mathrm{cl}\, R(f)$ holds.

pointwise self-homeomorphic

A topological space X is called *pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a set V such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *pointwise self-homeomorphic* if it is pointwise self-homeomorphic at each of its points.

pseudo-arc

A *pseudo-arc* is the only (up to homeomorphism) hereditarily indecomposable arc-like continuum.

pseudo-confluent

A mapping from X to Y is *pseudo-confluent* if every irreducible continuum in Y is the image of a continuum in X.

property of Kelley

A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see e.g. [64, Definition 16.10, p. 538]).

quasi-monotone

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]) *quasi-monotone* provided that for each subcontinuum Q of Y with the nonempty interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q.

radially convex at

Given a continuum X with an arc-structure A and a point $p \in X$, a metric d on X is said to be *radially convex at* p provided that $d(p, z) = d(p, y) + d(y, z)$ for every points $y, z \in X$ with $y \in A(p, z)$ (see [42, I.4, p. 551]).

ramification point

Point of order at least 3 is called *ramification point* of X; the set of all ramification points of X is denoted by $R(X)$.

rational

A continuum is *rational* if every two of its points can be separated by a countable point set.

real curve

A continuum is a *real curve* if it is a continuous 1-1 image of the real line.

recurrent point of

Let a continuum X and a mapping $f : X \to X$ be given. For each natural number n denote by f^n the n-th iteration of f. A point $p \in X$ is called a *recurrent point of* f provided that for every neighborhood U of p there is $n \in \mathbb{N}$ such that $f^n(p) \in U$. The set of recurrent points of a mapping $f : X \to X$ are denoted by $R(f)$.

refinable (monotonely)

A map r from a compact metric space X onto a compact metric space Y is *(monotonely) refinable* if, for each positive number ε, there is a (monotone) ε-map f from X onto Y such that, for each x in X,

$$d\left(f\left(x\right), r\left(x\right)\right) < \varepsilon.$$

regular

If X is a continuum and $p \in X$, then X is said to be *regular at* p provided that there is a local base $\mathcal{L}_p$ at p such that the boundary of each member of $\mathcal{L}_p$ is of finite cardinality. A continuum is said to be *regular* provided that X is regular at each of its points.

A continuum is *Regular* if each two of its points can be separated by a finite point set.

retract, retraction

Let X and Y be continua. A mapping $f : X \to Y$ is said to be a *retraction* provided that $Y \subset X$ and the restriction $f|Y : Y \to f(Y) \subset X$ is the identity; then Y is called a *retract* of X.

rigid

A nondegenerate topological space X is said to be *rigid* if it has a trivial autohomeomorphism group, i.e., if the only homeomorphism of X onto X is the identity.

selectible

A continuum X is said to be *selectible* provided that there exists a mapping $\sigma : C(X) \to X$ (called a *selection* for $C(X)$) such that $\sigma(A) \in A$ for each continuum $A \subset X$ (see e.g. [64, p. 253]).

scattered

A point set X is *scattered* if every subset Y of X has a point that is not a limit point of Y.

self-homeomorphic

A topological space X is called *self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ such that V is homeomorphic to X.

semi-aposyndetic

A space X is *semi-aposyndetic* if, for each two of its points, X is aposyndetic at one of them with respect to the other.

semi-confluent

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]):
– *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

semi-continuum

A *semi-continuum* is a continuumwise connected point set.

semi-continuous

Let a continuum X, a compact space Y and a function $F : X \to 2^Y$ be given. Put

$$F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

The function F is said to be *lower (upper) semi-continuous* provided that $F^{-1}(B)$ is open (closed) for each open (closed) subset $B \subset Y$. It is said to be *continuous* provided that it is both lower and upper semi-continuous. This notion of continuity agrees with the one for mappings between metric spaces.

semi-locally connected

A continuum M is *semi-locally connected* at the point p if, for every domain D containing p, there exits a domain E lying in D and containing p such that $M \setminus E$ has only a finite number of components. A continuum is *totally non-semi-locally connected* if it is not semi-locally connected at any point.

simple closed curve

A *simple closed curve* is any space which is homeomorphic to the unit circle.

simple $\mathfrak{m}$-od

Let $\mathfrak{m}$ be a cardinal number. By a *simple* $\mathfrak{m}$-*od* with the center p we mean the union of $\mathfrak{m}$ arcs every two of which have p as the only common point.

simple triod

The union of three arcs emanating from a point v is called a *simple triod* provided the singleton v is the intersection of any two of the arcs ; the point v is then called the *top* of the simple triod, and the arcs are called its *arms*.

shore point

A point p of a dendroid X is called a *shore point* of X if there exists a sequence of subdendroids X_n of $X \setminus \{p\}$ such that $X = \operatorname{Lim} X_n$. A shore point of X that is not an end point of X is called an *improper shore point* of X (see [71, p. 939]).

smooth (dendroid)

A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each point $x \in X$ and for each sequence of points $\{x_n\}$ in X tending to x, the sequence of arcs px_n tends to the arc px. A dendroid X is said to be *smooth* provided that it is smooth at some point $p \in X$ (see [34, p. 298]).

strongly chaotic

A nondegenerate topological space X is said to be *strongly chaotic* if for any two distinct points p and q of X there exist open neighborhoods U of p and V of q respectively such that no open subset of U is homeomorphic to any subset of V;

strongly pointwise self-homeomorphic

A topological space X is called *strongly pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a neighborhood V of x such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *strongly pointwise self-homeomorphic* if it is strongly pointwise self-homeomorphic at each of its points.

strongly rigid

A nondegenerate topological space X is said to be *strongly rigid* if the only homeomorphism of X into X is the identity of X onto itself.

strongly self-homeomorphic

A topological space X is called *strongly self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ with nonempty interior such that V is homeomorphic to X.

strongly unicoherent1

A continuum X is *strongly unicoherent* provided X is unicoherent and each proper subcontinuum with interior is unicoherent.

strongly unicoherent2

A continuum X is said to be *strongly unicoherent2* provided that it is unicoherent and for each pair of its proper subcontinua A and B such that $X = A \cup B$, each of A and B is unicoherent (see [2, p. 587]).

Suslinean

A continuum is *Suslinean* if it does not contain uncountably many mutually exclusive nondegenerate subcontinua.

$T_A(x)$

For each point x of a continuum X with an arc-structure A we define (see [42, I.3, p. 550]) $T_A(x) = \{y \in X$: each convex subcontinuum of X with y in its interior contains $x\}$. Since $T_A(x)$ is always closed, we have $T_A : X \to 2^X$.

terminal continuum

A proper subcontinuum K of a continuum X is said to be a *terminal continuum* of X provided that if whenever A and B are proper subcontinua of X having union equal to X such that $A \cap K \neq \emptyset \neq B \cap K$, then either $X = A \cup K$ or $X = B \cup K$.

thin

A topological space S is *thin* if and only if for each two homeomorphic subset A and B of S, there is a homeomorphism h of S onto itself such that $h(A) = B$.

translation

If P and Q are subspaces of a metric space X with a metric d, and ε is a positive number, then a mapping $g : P \to Q$ is called an *ε-translation* provided that $d(p, g(p)) < \varepsilon$ for each point $p \in P$.

tree

A *tree* is a graph which contains no simple closed curve. A tree as a one-dimensional acyclic connected polyhedron, i.e., a dendrite with finitely many end points.

triod

A continuum is called a *triod* provided it contains a subcontinuum whose complement is the union of three nonempty pairwise disjoint open sets.

unicoherent

A connected topological space S is said to be *unicoherent* provided that whenever A and B are closed, connected subsets of S such that $S = A \cup B$, then $A \cap B$ is connected.

Let a continuum X and its subcontinuum Y be given. Then X is said to be *unicoherent at* Y provided that for each pair of proper subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B \cap Y$ is connected (see [72, p. 146]).

uniquely arcwise connected

A continuum is said to be *uniquely arcwise connected* provided that for every two of its points there is exactly one arc in the continuum joining these points.

universal

Let a class $\mathcal{S}$ of spaces be given. A member U of $\mathcal{S}$ is said to be *universal for* $\mathcal{S}$ if every member of $\mathcal{S}$ can be embedded in U, i.e., if for every $X \in \mathcal{S}$ there exists a homeomorphism $h : X \to h(X) \subset U$. Accordingly, a dendrite is said to be *universal* if it contains a homeomorphic image of any other dendrite.

weak cut point

A point x of a continuum M is a *weak cut point* of M if there are two points p and q of M such that every subcontinuum of M that contains both p and q also contains x.

weakly chainable

A continuum is *weakly chainable* if it is a continuous image of a chainable continuum.

weakly confluent

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]) *weakly confluent* provided that for each subcontinuum Q of Y there is a component the set $f^{-1}(Q)$ which is mapped under f onto Q.

weakly hereditarily unicoherent

A continuum is said to be *weakly hereditarily unicoherent* if the intersection of any two of its subcontinua with nonempty interiors is connected (see [72, p. 152] and references therein).

weakly monotone

A surjective mapping $f : X \to Y$ between compact spaces is said to be (see [59, Chapter 3 and 4, p. 12-28]) *weakly monotone* provided that for each subcontinuum Q of Y with the nonempty interior each component the set $f^{-1}(Q)$ is mapped under f onto Q.

weakly smooth at a point

A dendroid X is said to be *weakly smooth at a point* $p \in X$ provided that the subspace of 2^X consisting of all subarcs of X of the form px for $x \in X$ is compact (see [57, p. 113]).

weakly Suslinian

A continuum X is said to be *weakly Suslinian* provided that X is not the union of more than one pairwise disjoint nondegenerate subcontinua of X.

widely connected

A nondegenerate connected space X is *widely connected* if each nondegenerate connected subset of X is dense in X.

width $w(X)$

For any compact metric space X, the *width* $w(X)$ of X is the l.u.b. of the set of all real numbers a which satisfy the following condition: for each $\varepsilon > 0$, there exists a finite open cover C of X such that $meshC) < \varepsilon$ and for each chain C' which is a subcollection of C there is a member A of C such that $d(A, C'^*) \geq a$.

Appendix B

List of examples

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